The Statistical Mechanics of Travelling Salesman type Problems

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In the standard travelling salesman problem (TSP), one is given a list of N cities, and an $N \times N$ distance matrix $\mathbf{D}$, where $\mathbf{D}_{ij}$ is the distance between the ith and jth cities in the list, and the problem is to find the shortest circuit that visits each city at least once. In a different version of the problem, one asks for the number of different circuits that have a total length lying betwen two given values. This problem is well-known to be in the so-called NP-complete class, and the best-known algorithm to find the optimal path takes a time that increases exponentially with N.

In a statistical mechanical approach to the problem, the distance matrix $\mathbf{D}$ is drawn at random from a prescribed ensemble, and one asks for the probability distribution of the length of the shortest path, or in the variation, of the number of paths having length lying between two prescribed limits. This problem then can be interpreted as finding the number of states of a system lying between two values of energy, in a system with quenched disorder, provided by the randomly chosen distance matrix $\mathbf{D}$, with the allowed states of the system being the $(N-1)!$ possible circuits.

In the Euclidean TSP corresponds to a particular choice of the ensemble of the distance matrices $\mathbf{D}$. One takes a unit square, say with periodic boundary conditions, and places N points on it random, each independently of others, with a prescribed, say uniform, probability density. The distance $\mathbf{D}_{ij}$ is assumed to be the euclidean distance between the points i and j.

It is easy to see that for a randomly chosen circuit, the average length is proportional to N. The average distance between nearest neighbors scales as $N^{-1/d}$ in d dimensions, and hence the typical length of the shortest circuit varies as $N^{1-1/d}$, for large N. Dean et al do not study the shortest circuit. In effect, they determine the asymptotic number of paths that have a length less than ϵN, for all values of ϵ. In the limit of large N, the fractional number of all circuits having total length between $N\epsilon$ and $N\epsilon + dE$ varies as $\exp(NS(\epsilon))dE$, and they are able to determine exactly the function $S(\epsilon)$. Actually, Dean et al determine ϵ as a function of the "temperature", which is the thermodynamical conjugate variable to the entropy S.

To find the solution, Dean et al convert the problem with a particular realization of cities as a constraint on the density of sites visited by a random walker. This constraint is taken care of by a Lagrange multiplier, and the problem is converted to that of a random walker with a distribution of step sizes, moving in an external potential, which is constructed so that the spatial density of sites visited by the walker equals the density of cities. The authors have verified the results of their analytical calculation by Monte Carlo simulations. They have also studied negative temperatures, which in the limit of large temperatures is the longest path problem.

There are only a handful of problems with quenched disorder that have been solved exactly. It would be interesting to see if one can extend this technique to find solutions for other problems with quenched disorder.