The structure of foam cells: Isotropic Plateau polyhedra
Europhys. Lett. **67** (2004) 484
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and

Analysis of 3-D network structures
Philosophical Magazine **85** (2005) 3
Author: M.E. Glicksman

and

Structure of Random Foam
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Recommended and a commentary by Randall D. Kamien, University of Pennsylvania

In the first two of these three papers, the authors have developed a mean-field theory for random foams in three dimensions. The last paper, through numerical work relying on Surface Evolver [1], confirms these mean-field results through the study of randomly assembled foams of 1728 bubbles. To get a sense of the difficulty in studying random foams, it is instructive to consider the problem of ordered foams, those with their bubbles arranged into a lattice. In the wet limit, the bubbles are spherical and the problem is closely related to that for packing of spheres with harmonic contacts [2]. In the dry limit, the bubbles necessarily distort from their ideal spherical shape and become polyhedra. Even in the special case of lattices, finding the properties of different foams is notoriously difficult. For instance, in 1887 Kelvin posed the question of how to fill space with polyhedra of equal volume so as to minimize the total area of the faces [3]. Even if one were to restrict the search over the simplest periodic structures representing the 230 space groups, the combinatoric problem is foreboding – though the volume of each cell is fixed, the topology is not. The number of edges and faces can change as long as Plateau's rules are satisfied locally (*i.e.* all vertices in the foam are connected to exactly 4 edges pointing towards the corner of a tetrahedron, and every edge belongs to 3 faces with 120° dihedral angles). The resulting combinatoric possibilities are many and, indeed, it took over 100 years for Kelvin's proposal (the BCC lattice) to be outdone by Weaire and Phelan's A15 lattice – a foam with two *different* unit cells of equal volume but differing topology [4]. How then to study random foams where the arrangement of the bubbles can vary too?

In the first two papers, the authors first consider polyhedra with F faces. Because Plateau's rule requires that each vertex in the foam is 4-fold coordinated, it follows that each vertex on every polyhedral cell is the corner of 3 faces. Letting η be the number of vertices (or edges) per face, the total number of vertices on a single polyhedron is $V = F\eta/3$, the number of edges $E = F\eta/2$. The Euler character ($V - E + F = 2$) implies that $\eta = 6 - 12/F$. For η to be integral, $F = 2, 3, 4, 6, 12$, or ∞. This leaves three degenerate polyhedra $F = 2, 3$, and ∞, and three well-known polyhedra, the tetrahedron, the cube and the dodecahedron (shown in Figure 1 of the first paper and Figure 5 of the

second paper). Imagining these polyhedra in a foam, the faces must puff out in order to satisfy Plateau's rules explained above. The authors show that the faces are spherical caps and, through judicious use of the Gauss-Bonnet theorem, develop formulae for the volume Vol, the specific area $e \equiv A(\mathrm{Vol})^{-2/3}$, and specific edge length $L(\mathrm{Vol})^{-1/3}$ versus F. At this point, the authors relax any constraints on F and allow the number of faces to be arbitrary, now viewing F as an average number of faces per cell in a foam. Thus, by using these isotropic Plateau polyhedra (IPP) or average $\mathcal{N}$-hedra (ANH), they are able to characterize mean properties of foams where each polyhedron is forced to satisfy the rules of local mechanical equilibrium. These results can then be used to study models of growth and coarsening and can be compared to other approximate theories. It is somewhat surprising that the specific area of the ANH/IPP bubbles is relatively independent of F, especially when compared to the specific area of polyhedra with *flat* faces.

In the third paper, some of the same authors do a large-scale study of random foams. Brakke's Surface Evolver [1] can find the areas and edge lengths in foams given a particular lattice and a particular bubble topology. Using new algorithms implemented by Brakke, the authors were no longer restricted to fixed bubble topology and a true minimization could be performed. In their numerics, they verify the independence of the specific area on the average value of F, corroborate the numerical value obtained in the previous paper, confirm the scaling of the specific length with F, and agree with the heroic, seminal experimental work of Matzke [5]. In addition, the third paper studies the effect of volume polydispersity on the total foam energy (*i.e.* area). Interestingly, polydispersity has a bigger effect on the number of faces per cell than on the number of edges per face, driving the former from $\langle F \rangle = 13.7$ down to about 7 faces per cell. The numerical results show a monotonically decreasing area as the polydispersity grows, in agreement with previous work and expectations. It is interesting to compare this to the effect of polydispersity on hard sphere packing: for hard spheres, polydispersity only slightly increases the packing fraction of random structures and decreases the packing fraction of ordered lattices [6]. Similarly, it is known that changing the hard spheres into hard ellipsoids can increase the volume fraction [7]. The work on foams adds another piece to the puzzle of how to describe and quantify random packing.

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