First passage times in complex scale-invariant media

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http:arXiv:0711.0682
Nature **450**, 77-80 (2007)

Recommended with a Commentary by Bernard Nienhuis, University of Amsterdam

One of the classical problems in stochastic processes is characterization of the first-passage time (FPT), the stochastic variable equal to the time that a Markov process takes to meet a particular final condition for the first time. The full characterization[2] is the probability distribution of this FPT. A popular measure is its mean value, the MFPT[1]. Since the probability distribution of the FPT has a power law tail, the MFPT is not for all purposes the relevant quantity. The FPT has many applications, such as the question how likely it is that I will be affected by a particular transmittable disease in my life time, that the Dow Jones index decreases by 30% in 2008, or the time it takes for droplet to grow beyond a critical size.

Since the FPT obviously does not depend on what happens after the final condition is met, for all practical purposes we may terminate the process at that time, and thus treat the final condition as a set of states with zero exit probability, i.e. absorbing states.

For Brownian motion in Euclidean space or regular lattices, as well as a variety of other cases[3], many FPT distributions, or MFPT's can be calculated in closed form. The authors of this paper consider random walks in complex but scale-invariant media[4], possibly themselves defined stochastically. They use the language of graphs[1], and consider classes of networks, such as scale-free networks and small-world networks. A different name, exit time or $t_{\rm exit}$, is reserved for the time it takes for a walker to come for the first time at a certain distance from its starting point. In scale invariant systems the exit time increases with the distance as $t_{\rm exit} \propto r^{d_{\rm w}}$, where $d_{\rm w}$ is called the walk dimension. Another dimension that plays a major role is the fractal (Hausdorff) dimension of the network, defined by the number of nodes within a sphere of radius r: $N \propto r^{d_{\rm f}}$

The result of the paper can be stated as follows. The authors consider the first time that a walker visits a specific site at a distance r from the starting point, in a finite domain of N sites. They state that the MFPT varies with

r and N as

$$\begin{aligned} N(A - Br^{d_\mathrm{w}-d_\mathrm{f}}) &\quad \text{for} \quad d_\mathrm{w} < d_\mathrm{f} \\ N(A + B\log r) &\quad \text{for} \quad d_\mathrm{w} = d_\mathrm{f} \\ N(A + Br^{d_\mathrm{f}-d_\mathrm{w}}) &\quad \text{for} \quad d_\mathrm{w} > d_\mathrm{f} \end{aligned}$$

The constants A and B depend on the characteristics of the network, but not on the geometry of the confining domain. A slight modification is necessary in scale free networks, i.e. networks in which the number of nodes, n, with given number of links, ℓ, decreases as a power of ℓ. In this case the formal d_f is infinite. However, another fractal dimension, the box dimension d_b is still well-defined and finite, if the network can be covered with N_b boxes of (linear) size ℓ_b and $N_\mathrm{b} \propto N\ell_\mathrm{b}^{d_\mathrm{b}}$. For these networks, the same formulae for the MFPT hold, with d_f replaced by d_b.

These results which are very interesting and far reaching, depend on the assumptions (i) that the Greens function in a finite domain is well approximated by its value for the infinite domain, and (ii) that it has a scaling form.

In my opinion the result is an important step forward in the field of transport in random media, in particular the result is quite specific. However, there is also room for criticism, on the following points.

(i) The main text speaks mainly in the language of networks, which usually do not have an embedding metric. Therefore distance is always defined as the minimal number of steps that connect two points. In the simulations, however, distance is defined in terms of the embedding metric. It is not clear what distance should be used.

(ii) In the literature the FPT is used in a more general sense[1] than it is used here, the FPT to a single site. In particular the exit time, which the authors explicitly exclude, is a good example of an FPT. In continuous spaces the FPT to a single site is always infinite. Moreover, for walks on a graph, it is just a matter of definition: any termination condition can be defined as a single node, so that there is no formal distinction between the FPT and the exit time.

(iii) A minor point: the paper claims that the MFPT (in their restrictive definition) is always infinite in an unbounded domain. Exceptions to this are the half infinite chain, as well as the infinite chain with a drift towards the origin.

References

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