Some New Exact Relations in Strongly Interacting Fermi Gases

(1) *Energetics of a strongly correlated Fermi gas, Shina Tan, Annals of Physics 323, 2952-2970 (2008), arXiv:0505200*
(2) *Large Momentum part of fermions with large scattering length, Shina Tan, Annals of Physics 323, 2971-2986 (2008), arXiv:0508320*
(3) *Exact Relations for a Strongly Interacting Fermi Gas from the Operator Product Expansion, Eric Braaten and Lucas Platter, arXiv:0803.1125, Physical Review Letters 100, 205301 (2008)*
(4) *Universal properties of the ultracold Fermi gas, Shizhong Zhang and Anthony Leggett, arXiv:0809.1892, Physical Review A 79, 023601, (2009)*

Recommended with a Commentary by Tin-Lun Ho, Ohio State University, Columbus, Ohio

Dilute Fermi gas is one of the first problems tackled in the early days of quantum many-body theory. Many renowned scientists, including Galitskii, Lee, Huang, Yang, and Gell-Mann had worked on it. Most studies in those days focussed on the weak-coupling limit ($n^{1/3}a << 1$), where n is the density of the gas, and a is the scattering length of the two-body potential. In the last five years, development of methods to cool atoms to ultra-cold temperature has led to considerable interest in quantum gases in the strongly interacting regime ($n^{1/3}a >> 1$). In this limit, a robust fermion superfluid is found, with T_c/T_F (~ 0.5), the highest ratio ever known. This superfluid also exhibits scale invariance and hydrodynamic behavior similar to that in heavy ion collision experiments. It undergoes BEC-BCS crossover as scattering length is varied, and in case of unequal spin population, its phase diagram is believed to have features similar to those of quark matter.

Mathematically, strongly interacting Fermi gas is a very challenging problem. The reason is that as scattering length $a \to \pm\infty$, interaction scale drops out from the problem. Perturbative methods are inapplicable and it is hard to estimate the errors of well known approximation schemes. Faced with such difficulty, any analytic results about strongly interacting Fermi gases is highly valuable.

About four years ago, in a sequence of original papers, Shina Tan worked out a number of exact relations for strongly interacting Fermi gases with large scattering lengths. The earliest two of these papers are listed above. Among these exact relations, two are particularly fundamental. The first, which I shall call Energy Relation, relates the energy of dilute Fermi gas (with arbitrary spin polarization) to the momentum distribution $n_{\mathbf{k}.\sigma}$,

$$E = \sum_{\mathbf{k},\sigma} (\hbar^2 k^2/2m) \left(n_{\mathbf{k},\sigma} - C/k^4 \right) + \hbar^2 \Omega C/(4\pi a m) \tag{1}$$

where Ω is the volume. The quantity C, referred to as the contact density by Tan, is the large momentum limit of the momentum distribution, $C = \mathrm{Lim}_{k\to\infty} k^4 n_{\mathbf{k},\sigma}.$; and is independent of spin σ. The second, referred by Tan as Adiabatic Theorem, shows the rate of change of energy at constant entropy is also given by C .

$$\left(\frac{\mathrm{d}E}{\mathrm{d}(-1/a)} \right)_S = \frac{\hbar^2 \Omega C}{4\pi m}. \tag{2}$$

These two relations show that the behavior of momentum distribution at momenta far *above* the Fermi surface plays a central role in determining the total energy of the system. Such behavior is captured by the contact coefficient C, which describes the probability of two particles coming close together, and is dependent on the state of interest, i.e. superfluid or normal.

Tan's derivations of eq.(1) and (2) were very unconventional and quite involved. As is not unusual for new and interesting developments, the scientific journals did not publish Tan's work for four years. It's long due recognition has come only lately from the rederivation of the results by Braaten and Plaater, who showed that Eq.(1) and (2) can be obtained using the method of operator product expansion invented by Ken Wilson in the 70s to handle nuclear forces. Some of Tan's results, such as the Adiabatic Theorem, were also derived independent by Shizhong Zhang and Anthony Leggett in a paper in September 2008, apparently unaware of Tan's work. The derivation of Zhang and Leggett is very intuitive, shedding new light into the Adiabatic Theorem. In June 2008, at a conference in Copenhagen on Strongly Interacting Gases and Quark-Gluon Plasma where Tan presented his results, the cold-gas community began to realize that the contact density C is of great relevance to many experiments, linking many measurements that were thought to be unrelated together.

Tan's results apply to the low density regime with short-range interactions, in which the range of interactions r_o is much smaller than the inter-particle separation $n^{-1/3}$, where n is the density. It is effectively an expansion about both the limits $n^{1/3}r_0 = 0$ and $1/(n^{1/3}a) = 0$. In this limit, the many-body wave function has the following general property: as two fermions with opposite spin approach each other, the many-body wavefunction Ψ as a function of their separation r reduces to the solution of two-body Schrodinger equation as $r \to 0$, and therefore has the general form $\Psi \propto (1/r - 1/a)$ in 3D. This means the Fourier transform of Ψ is k^{-2} as $k \to \infty$, and $n_{\mathbf{k}} \sim k^{-4}$ in the limit of large k . This is problematic because it implies a linear divergence of kinetic energy $\sum_{\mathbf{k}} k^2 n_{\mathbf{k}}$ in the large momentum cutoff. Such divergence, of course, is not real, as the behavior has to be cutoff eventually at some high momentum scale. A cutoff independent way to eliminate this divergence is to subtract off the divergent piece explicitly, which is the first term of Eq.(1). What is not obvious is that the left over piece is also given by the contact density C.

The relations of Tan also raise new questions. The structure of Eq.(1) reminds one of Landau Fermi liquid theory, where energy changes are related to changes in $n_{\mathbf{k}}$ through the Landau parameters. At present, how the Energy Relation is related to Landau Fermi liquid theory is not clear. The short distance behavior $(1/r - 1/a)$ so crucial in Tans proof also shows that his results will fail in the limit $a \to 0$, where asymptotic form $(1/r - 1/a)$ becomes ill defined. The correct expression of total energy at $a = 0$ is a fundamental problem, and a generalization of the exact results of Tan to cover all scattering lengths is needed.

Finally, I would like to thank Chandra Varma for many very useful comments on the first draft of this Commentary.