On the granular stress-geometry equation

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Europhysics Letters 105, 28001 (2014)

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Traditional theories of elasticity assume the existence of an ordered reference state, about which the system deforms under the application of external stress. One can then calculate the stress from a constitutive relation between stress and strain, usually derived from an energy functional. Unfortunately, such an approach breaks down for amorphous materials, simply because one cannot define a suitable ordered reference state. In such systems, one can attempt to construct a theory starting from the conditions of mechanical equilibrium:

$$\vec{\nabla} \cdot \hat{\sigma} = 0, \quad \hat{\sigma} = \hat{\sigma}^T \tag{1}$$

In 2D, these provide two equations and three independent components of the stress tenor, $\hat{\sigma}$, and in 3D there are three equations for six independent components. In elastic materials, the “missing” equations needed to determine the stress tensor are provided by imposing the constraint that the stress tensor is linearly related to the strain tensor. One then obtains a compatibility condition based on the observation that the strain tensor is constructed from the derivatives of a vector field: the displacements from a reference structure. It can be shown[1] that these compatibility conditions are enough to close the problem: they provide the missing equations relating components of the stress tensor. Since this approach does not rely on *knowing* a reference structure but only the elastic moduli, it can in principle be applied to both amorphous and crystalline solids. The theoretical difficulty that we are faced with is computing the elastic constants from microscopic interactions, and a lot of recent work has focused on this aspect.

The theoretical difficulty that we face in granular materials is much more fundamental since there is an indeterminacy of contact forces at the microscopic level, since the tangential forces satisfy only an inequality condition for grains with friction coefficient μ : $f_t \leq \mu f_N$ (Coulomb inequality). In principle then, one needs to know the full packing history to be able to determine the configuration of forces. There are special states, however, where the constraints of force and torque balance uniquely define the contact forces for a given packing geometry. For these, so called isostatic states, there has been a search for stress-only closure schemes for static granular equilibria: schemes in which the “missing” equations can be obtained from the force and torque balance constraints alone[2]. In spite of work spanning two decades, the issue of constitutive relations and the precise nature of the stress-only description has remained elusive.

In a recent paper, DeGiuli and Schoof construct a mean-field picture involving the stress tensor and the fabric tensor. The fabric tensor represents the “fabric” of the geometric contact network, and can be constructed from a knowledge of the grain positions. DeGiuli and Schoof show that, the fabric tensor and the deviatoric part of the stress tensors are coupled, and the resulting equation provides the closure needed to determine all components of the stress tensor. So, given a fabric tensor, one can uniquely determine all the components of the stress tensor. The relationship they deduce crucially relies on imposing all of the constraints of mechanical equilibrium: force and torque balance at each and every grain. The DeGuili-Schoof framework reformulates the stress transmission problem in terms of a vector and a scalar field defined on the voids surrounding the grains, which are gauge fields that impose these constraints. These fields allow us to formulate the problem in terms of geometric structures in a dual space, which provides a novel approach to the statistical mechanics of granular media. This dual space approach is being increasingly adopted to describe the mechanical properties of granular solids, [3–5] and has offered new ways of understanding these materials.

A contentious issue surrounding the nature of stress transmission in granular materials has been the mathematical form of the continuum equations: are they elliptic or hyperbolic ? The results of Deguili and Schoof demonstrate that the answer depends on the nature of the fabric tensor: if the fabric tensor is spatially homogeneous then the equations are elliptic whereas if there are strong spatial gradients in the fabric, then the equations are nearly hyperbolic. In our view, this provides an elegant answer to a long-standing question.

The approach of DeGiuli and Schoof solves the problem of determining a constitutive relation for static granular assemblies in an elegant manner: it provides a stress-geometry equation that is to be solved, subject to the Coulomb inequalities, for a given fabric tensor and boundary values of stresses. Their derivation relies on a decoupling of stresses and geometry on a mesoscopic scale that is small compared to length scales characterizing stress gradients. It will be instructive to test this assumption in experiments and numerical simulations.

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