The simplest model of jamming

Silvio Franz and Giorgio Parisi, arXiv:1501.03397 [cond-mat.stat-mech],
Recommended with a commentary by Robijn Bruinsma (UCLA)

The packing fraction ρ of a collection of hard spheres in three dimension cannot exceed a maximum of about 0.74. In that limit, the spheres are arranged in a close-packed crystalline lattice (such as the HCP lattice). If, however, a loose collection of hard spheres is compactified, starting from some random initial condition, then the maximum packing fraction ρ_c that can be achieved by compaction is well below 0.74: the system *jams* during compaction before it can crystallize. In fact, ρ_c is always less than about 0.64, known as the random close packing limit. That does not mean that there are no arrangements of hard spheres with packing fractions in between 0.64 and 0.74. It is in fact straightforward to construct such arrangements [1]. What it does mean is that hard sphere assemblies with packing fraction between 0.74 and 0.64 are not accessible by compaction of randomly chosen sphere arrangements.

The study of the jamming transition has made great strides in recent years [2]. It is now understood that the jamming of a collection of hard objects corresponds to a *critical point*. Just as for other critical systems, a correlation length can be defined that diverges like a power law when ρ approaches ρ_c, much like the correlation length of the Ising model diverges when the temperature approaches the critical tempearture. Unlike the critical exponents of the Ising model, numerical simulations indicate that certain critical exponents of jamming are independent of the spatial dimension ("super-universality"). The concept of jamming has been extended to include soft objects [3] - even living cells - and it has been proposed that jamming could be a paradigm for the finite-temperature formation of glasses [2].

James Clerk Maxwell first observed that for a system of hard objects to be fully constrained, the number of constraints imposed by the excluded volume interaction must exceed the number of degrees of freedom. That leads to the condition that at the onset of jammimg the mean number of contacts Z between a sphere and its neighbors must equal twice the spatial dimension of the system. Sphere arrangements that obey this condition are called *isostatic*. In the theory of computation, problems that require maximizing a quantity (such as the packing density of a hard sphere system) that depends on a certain number of degrees of freedom subject to a certain number of linear inequalities (such as the excluded volume conditions) are known as a *linear programming problems*. Linear programming problems are often encountered in management and planning problems and have been studied extensively. It has been noted, for a number of NP-complete computational problems of this type, that if one varies the ratio of the number of constraints over the number of variables then the point separating the under-constrained regime from the over-constrained regime has the character of a critical point that has power-law type singular properties [4]. These are known as "SAT-UNSAT" critical points. The paper of Franz and Parisi suggests that it is productive to view jamming as a SAT-UNSAT transition.

Franz and Parisi start from a toy model for jamming in high dimensions where M points ξ_i^μ - the obstacles - are randomly distributed over the surface of an N-dimensional sphere with radius $\sqrt{N}$, with i running from 1 to N and μ running from 1 to M. One free particle is added with coordinates x_i that has to satisfy the condition $|\xi^\mu - \mathbf{x}| > \sigma$, with σ playing the role of the hard-core radius. As the ratio $\alpha = M/N$ increases, the solution space shrinks to zero, which they identify as jamming point. Next, they observe that this toy problem is a special case of the *perceptron*, a machine-learning algorithm with a long history of applications in linear programming type problems. It also has been applied to implement the Hebb learning rules of neural networks. The perceptron algorithm requires the determination of a space of vectors x_i such that the M "gap" quantities $r_\mu = N^{-1/2} \sum_i \xi_i^\mu x_i - \kappa$ are all positive. In machine learning, κ is known as the bias. By squaring the conditions $|\xi^\mu - \mathbf{x}| > \sigma$ for the toy model, it is easy to see that the toy model is a perceptron with $\sigma^2 = 2N + \kappa$. It is however productive to treat κ as a free parameter. Franz and Parisi find - analytically - that for positive κ the system is not jammed while for negative κ there is a jamming transition with critical exponents that coincide with those found earlier for jamming in high dimensions.

The importance of the Franz-Parisi toy model/perceptron is that it links isostaticity and jamming to *replica symmetry breaking*, the breaking up of configuration space into separate domains. The jamming line they find lies in a region of the phase diagram with broken replica symmetry. In fact, they find that the condition of isostaticity is *not* an essential condition for the appearance of critical jamming behavior but that replica symmetry breaking is. The paper of Franz and Parisi implies that the isostaticity criterium for jamming should be abandoned in favor of replica symmetry breaking! That might be welcome news for experimental tests of jamming because the concept of isostaticity is rather problematic for physical particles - such as colloids - whose surfaces are not mathematically smooth. It also is not clear how to apply isostaticity to particle systems at finite temperature. The concept of replica symmetry breaking applies, on the other hand, very well to finite-temperature systems (such as spin glasses) while it may be a unifying link between jamming and glass formation. Replica symmetry breaking also provides us with a nice visualization for the appearance of the packing structures with density higher than that of random close packing that are not accessible for arbitrary initial conditions.

[1] For example by assembling the spheres into larger spheres with a crystalline HCP structure and then randomly close-packing these larger spheres. Varying the ratio between the radii of the large and small spheres varies the packing fraction between the two limits.
[2] Andrea Liu and Sydney Nagel, Annu. Rev. Cond. Mat. Phys, 2010.
[3] Bi Dapeng et al. Soft Matter 10 1885-1890. 2014
[4] Ke Xu et al., Artificial Intelligence 171 514534, 2007