What is the fastest an egg (or anything else) can be scrambled?

- **A bound on chaos**
  J. Maldacena, S. Shenker, and S. Stanford, arXiv:1503.01409
- **A simple model of quantum holography**
  A. Kitaev, two KITP talks available online

Recommended with a Commentary by Joel E. Moore, UC Berkeley and LBNL

The concept that there might exist fundamental limits in many-particle quantum systems on the relaxation of a current, or more broadly the development of thermal equilibrium, has a long history. One extreme is that no thermalization occurs at all, as for example in a many-body-localized system. This commentary is about recent work that in rough terms is about the opposite limit: are there limits to how fast a system can achieve equilibrium or "chaos" (to be defined momentarily)? Recent work by Maldacena, Shenker, and Stanford [1] and by Kitaev (unpublished) answers this question in the affirmative and gives a specific bound on chaos that is saturated in one class of models.

Condensed matter interest in this question was stimulated by long-standing puzzles in transport in complex materials such as the normal state of cuprates and heavy fermion materials. (Since this is a brief commentary and not a review article, I will cite a few out of many works and the reader is encouraged to look at references therein for the full story.) These materials show anomalous, often linear, temperature dependences of the resistivity that do not appear to be controlled by disorder but rather by strongly correlated behavior of the electrons [2]. More quantitatively, many materials [3] show current relaxation times τ, inferred from measured conductivity, with

$$\tau \gtrsim \frac{\hbar}{k_B T}. \tag{1}$$

Infrequently, other power-laws than T^{-1} are observed, such as $T^{-3/4}$ in certain heavy-fermion compounds [4].

Some different forms of the bounded-relaxation concept in quantum many-body physics are as follows [5, 6, 7, 8]. The classic Ioffe-Regel criterion [5] in a metal is that the mean free path cannot meaningfully be shorter than a lattice spacing, but this depends on a quasiparticle scattering picture that is not obviously applicable in correlated systems. The specific form (1) was stated by Sachdev in the context of nonzero temperature above a quantum critical point [6], where the temperature provides the only natural scale that could determine relaxation, together with examples such as the superconductor-insulator transition of a bosonic model with particle-hole symmetry. A remarkable calculation using string-theoretic methods by Kovtun, Son, and Starinets [7] conjectured a general inequality involving the shear viscosity η and the entropy density s,

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}, \tag{2}$$

which applies in a wide range of fluids but has small violations in some theoretical large-N models. Introducing a velocity c and diffusion constant $D = c^2\tau$, which unlike the shear viscosity exists in $d = 1$, the viscosity bound (2) can be connected to the relaxation-time bound (1) [7, 8].

The new work is concerned not with relaxation but with the onset of chaos. In classical systems, chaos can be defined via exponential separation of trajectories: two initially similar states become very different at a later time, and the Lyapunov exponent gives the rate of growth of the separation. Naïve generalization to quantum mechanics does not work if we define the difference between two states as the overlap, since that is constant in time:

$$|\langle\psi_1(t)|\psi_2(t)\rangle| = |\langle\psi_1(0)|e^{iHt/\hbar}e^{-iHt/\hbar}|\psi_2\rangle(0)| = |\langle\psi_1(0)|\psi_2(0)\rangle|. \tag{3}$$

So one needs to find a different definition, and Ref. [1] looks at how the value of one observable V at time 0 influences another observable W at time t. This should be measured via some commutator of $V(0)$ and $W(t)$; the specific quantity that winds up being amenable to their analysis is

$$F(t) = \mathrm{Tr}\ [yV(0)yW(t)yV(0)yW(t)] \tag{4}$$

where the operator y corresponds to a $\pi/2$ rotation along the imaginary-time cylinder:

$$y^4 = \frac{e^{-\beta H}}{Z}. \tag{5}$$

Then $F(t) \to 0$ as $t \to \infty$ is a viable quantum definition of the so-called butterfly effect, the disappearance of the statistical influence of the earlier observable on the later one, and we label the time scale for F to vanish the scrambling time t_*.

There is a shorter time scale t_d, which the authors call a dissipation time, after which F takes a nonzero constant value F_d. Roughly, dissipation or relaxation corresponds to a weak form of thermalization, after which standard local observables typically relax close to their equilibrium values; scrambling or the butterfly effect is a very strong form of thermalization, after which two initial states are indistinguishable without measuring a macroscopic number of observables. [9] Kitaev emphasized in 2014 that the onset of scrambling and the butterfly effect will initially grow exponentially, in the sense

$$F_d - F(t) = \epsilon \exp \lambda_L t + \ldots, \tag{6}$$

and some large-N theories have small $\epsilon = 1/N^2$ and thus a parametric separation between dissipation and scrambling times. (A technical note: the derivation of the bound depends on large N and could have violations of order $1/N^2$, compared to violations of the viscosity bound (2) at order $1/N$.) The punchline of Ref. [1] is a specific bound on the Lyapunov exponent

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}. \tag{7}$$

The argument for this result is "elementary" in the sense of not requiring an AdS/CFT correspondence (basically, a dual representation of the problem as a gravity theory) or other technology, but depends on some fairly technical complex analysis in order to connect real-time dynamics with imaginary time where $\hbar/(k_B T)$ is a natural scale. It is worth pointing out that such bounds were motivated in part from observing that many models with gravity duals are "fast scramblers" (see [10] and other references in [1]) saturating the bound, and this may be another case, similar to entanglement entropy [11], where AdS/CFT correspondences lead to conjectures that turn out to be quite general and derivable by simpler methods.

A next question is how to find explicit examples where this bound is saturated. (Since presumably the dissipation time for any current should be shorter than the scrambling time, such bounds likely give bounds on current relaxation as well.) Two KITP talks by Alexei Kitaev, available online, give an answer by building a precise holographic version of a simplified version of the Sachdev-Ye model [12], which has random, infinite-ranged interactions that previously allowed some properties to be solved via a mean-field approach. A very recent preprint [13] shows that the zero-temperature entropy of both Kitaev's model and the Sachdev-Ye model is connected via holography to the famous Bekenstein-Hawking black hole entropy formula. Future work, one hopes, will connect the deep fundamental bounds with more realistic condensed matter Hamiltonians and ultimately explain the experimental observations. It is comforting to see that, along with the current interest in exotic equilibrium phases and far-from-equilibrium dynamics, there is significant progress on near-equilibrium problems of linear response and scrambling.

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