Measuring Entanglement by Swapping Quantum Twins

Featured paper:

Measuring entanglement entropy through the interference of quantum many-body twins, Rajibul Islam, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli, Markus Greiner. Nature 528, 77 - 83 (2015). arXiv:1509.01160.

Recommended with a commentary by Ashvin Vishwanath, UC Berkeley

Quantum entanglement, which is at the heart of quantum mechanical behavior, has become increasingly relevant to the study of many body systems. It has been extremely useful as a conceptual framework, and as a diagnostic tool in numerical simulation. For example, various quantum phases can be defined on the bases of their entanglement signatures, which allows for their classification. Also, since entanglement is responsible for the exponential complexity of quantum systems, its understanding has been important to efficiently simulate them. However, there have been fewer experimental attempts to measure quantum entanglement, particularly in ways that can be applied to many body quantum systems. The present reference takes a major step forward in this direction. In brief, using a pair of identical quantum states of bosons in a tunable optical lattice, the entanglement entropy of both the Mott and superfluid phases are measured. Although small systems are considered (4 sites, and subregions of size 2 sites), bigger sizes should be possible. This will bring many questions regarding entanglement entropy and its time evolution in many body quantum systems into the experimental realm.

Let us briefly recall some relevant definitions. Assume for a moment that the total system is in a pure state $|\Psi\rangle$ (the general case of mixed states requires minor modifications). Then, if we restrict attention to the subregion 'A', we define the density matrix ρ_A by tracing out all degrees of freedom outside, which we call region 'B'. That is :

$$\rho_A = \mathrm{Trace}_B |\Psi\rangle\langle\Psi|$$

Diagonalizing ρ_A yields probabilities p_i for the subsystem being found in different quantum states. In the absence of quantum entanglement between regions A and B, ρ_A also represents a pure state, so there is a single $p_1 = 1$ and all other probabilities vanish. More generally, in the presence of quantum entanglement we have at least a pair of nonvanishing p_i. A quantitative measure of entanglement hence needs to weigh the departure of ρ_A from a pure state. The most popular such measure is the von-Neumann entropy, defined as $S_A = -\mathrm{Trace}_A\left[\rho_A \log \rho_A\right]$ or equivalently:

$$S_A \;=\; -\langle \log \rho_A \rangle_{\rho_A}$$

this expresses the entanglement entropy as an expectation value. This brings us to the first problem with measuring entanglement entropy. Typically, if one is interested in the expectation value of an observable O defined in the region A one would calculate $\langle O \rangle_{\rho_A} = \mathrm{Trace}_A\left[\rho_A O\right]$. When measuring entanglement entropy, the operator to be measured is itself dependent on the state of the system i.e. $O \to \log \rho_A$. Therefore this measurement is qualitatively different from measuring a physical property like the local density of a system. A second problem is that when the size of the region 'A' is large, one is dealing with a non-local quantity, that needs to simultaneously utilizesinformation about the state of the system in an extended region. The second problem was solved using a quantum gas microscope, that provides a snapshot of the particle occupancy on all lattice sites.

Solving the first problem required several steps. First, one would like to convert the measurement into a more standard one that evaluates the expectation value of a fixed operator. For this it is convenient to move away from the von-Neumann entropy and use a different, but qualitatively similar measure of entropy, the Renyi entropy, S^A_n with an index $n > 1$, is defined as:$S^A_n = -\frac{1}{n-1} \log \mathrm{Trace}_A\left[\rho^n_A\right]$. Again, for a pure state $S^A_n = 0$, while for a mixed state $S^A_n > 0$, and hence it has the right qualitative properties to qualify as a measure of entanglement. In fact, in the limit $n \to 1$, it reduces to the von-Neumann entropy. A simple entanglement measure therefore is the $n = 2$ Renyi entropy:

$$e^{-S^A_2} = \mathrm{Trace}_A\left[\rho^2_A\right] = \langle \rho_A \rangle_{\rho_A}$$

In order to express this as an expectation value of a fixed operator, one considers a tensor product of the wavefunction with itself: $|\Psi\rangle \to |\Psi\rangle \otimes |\Psi\rangle$. That is, two identical copies of the wavefunction. Now, the second Renyi Entropy is expressible as the expectation value of an operator, called the Swap_A operator, which exchanges the state of the system in region A between the two copies, while leaving the 'B' degrees of freedom untouched. So: $\mathrm{Swap}_A|n_A,\, n_B\rangle|n'_A,\, n'_B\rangle = |n'_A,\, n_B\rangle|n_A,\, n'_B\rangle$. It is easily shown that:

$$e^{-S^A_2} = \langle\Psi|\langle\Psi|\mathrm{Swap}_A|\Psi\rangle|\Psi\rangle$$

thus we have to measure an expectation value of a fixed operator. The price we pay is that we need two identical copies of the state, for example by preparing two identical systems in their ground states.

Measuring the Swap operator is easily done in numerical simulations, where the above procedure is useful for extracting the entanglement entropy[1]. However to make it suitable for an experimental measurement, one needs a more physical observable. In fact, since the quantum gas microscope is designed to measure particle occupation number at a given lattice site, we want to reduce the measurement of the Swap operator, to one that measures site occupancies. This is done by constructing the many-body boson analog of the Hong-Ou-Mandel interferometer that is well known in the context of quantum optics [2].

The key observation is that the Swap operator, which exchanges particles (say bosons) in the two copies (copy 1 and copy 2), can be expressed very simply in terms of the sum and difference of boson operators. Boson operators a_{i1} and a_{i2} on site i in the two copies, these are exchanged by the Swap operator if site i is in region 'A'. Alternately, if we consider the sum and difference operators: $b_{i1} = \frac{a_{i1}+a_{i2}}{\sqrt{2}}$ and $b_{i2} = \frac{a_{i1}-a_{i2}}{\sqrt{2}}$, then the action of the Swap operator is to change the sign of the b_{i2} operators, when $i \in A$. This is implemented by the parity operator $P_i = (-1)^{n^b_{i2}}$ which measures the parity of the boson occupation numbers of the b_{i2} bosons. The total parity of these bosons in region 'A' : $P_A = \prod_{i\in A} P_i$ is directly related to the Swap_A operator and thus to the entanglement entropy:

$$\langle P_A \rangle = \langle \mathrm{Swap}_A \rangle = e^{-S_2^A}$$

Experimentally, one needs to transform the sum and difference operators, into site operators whose occupations can be measured. This is done by introducing a controlled tunneling J_y between the two copies for only those sites that reside in region A. Allowing the tunneling to act for a time $t = 2\pi/8J_y$ along with certain phase operations, leads to the desired transformation, so that ultimately the occupation numbers in the second copy exactly reflect the occupation numbers of the b_2 bosons, whose parity is then readily measured.

Experimental Results: In the featured reference, an optical lattice of size 4x2 was prepared in a Mott state with one bosonic atom (Rb_{87}) per site. The tunneling along the length 4 direction is controlled to tune the state between 'Mott' and superfluid phases, while the other direction represents the two identical copies. An additional complication of the experimental system is that the initial state is a thermal state and hence includes a thermal contribution

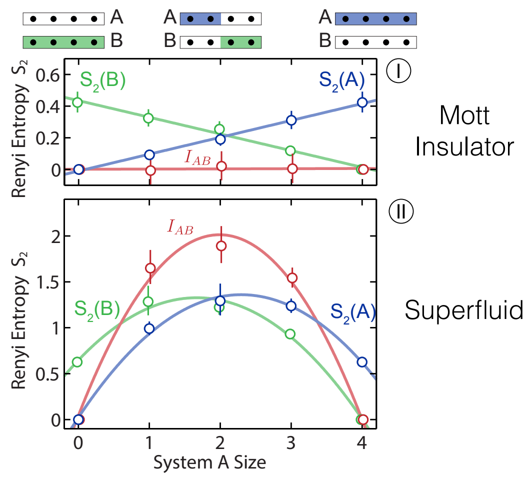

FIG. 1: Experimentally measured entanglement entropy of a 4 site system, with different sizes of subsystems 'A' (x-axis). Panel I: The system is in the Mott phase and the ground state is essentially a product state with little entanglement between subsystems. This is reflected in the mutual information I_{AB} (red curve) which remains small, independent of system size. Panel II: In the superfluid phase, where there is substantial entanglement due to the delocalization of bosons, the entropy is nonmonotonic in system size characteristic of the quantum entanglement contribution. I_{AB} (red curve) grows with system size and achieves a maximum when 'A' is half of the total system size, and then decreases. Figure taken from the featured reference.

to the entanglement entropy that scales linearly with system size. This is readily excluded by studying the mutual information $I_{AB} = S_A + S_B - S_{AB}$ which is sensitive to quantum entanglement between the subsystems A and B. The measured mutual information for the Mott and superfluid states are compared in the Figure (red curves). While the Mott state has negligible quantum correlations between sites, these are substantial for the superfluid state, and both agree with numerical simulations.

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[1] M. B. Hastings, R. Gonzalez, A. Kallin, R. Melko. Phys. Rev. Lett. 104, 157201 (2010).

[2] C. Moura Alves, and D. Jaksch, Phys. Rev. Lett. 93, 110501 (2004). A. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, Phys. Rev. Lett., 109, 20505, (2012).