

Universal entanglement dynamics

Quantum Entanglement Growth Under Random Unitary Dynamics

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Recommended with a commentary by Rahul Nandkishore, CU Boulder

Quantum entanglement is a phenomenon whereby distinct degrees of freedom exhibit ‘quantum correlations’ such that the wavefunction of the entire system cannot be expressed simply as a product of the wavefunctions of individual degrees of freedom. Quantum entanglement is emerging as a powerful unifying concept connecting diverse areas of theoretical physics. Studies of the *dynamics* of quantum entanglement have illuminated fields from many body localization [2] to holography [3]. A new preprint Ref.[1] now opens up a new line of attack in the study of entanglement dynamics.

A quantum system can always be divided into a subsystem (A) and ‘everything else’ (B). The reduced density matrix ρ_A of subsystem A, obtained by performing a partial trace of the full density matrix over the degrees of freedom in B, then contains all information that can be extracted from measurements made on A alone. Crucially, if regions A and B are entangled, this entanglement shows up in ρ_A . The degree of entanglement can be quantified by the Renyi entropies $S_n = \frac{1}{1-n} \log \text{Tr} \rho_A^n$, where $n = 0, 1, 2, \dots$ is the Renyi index. Here S_1 is understood as the Von Neumann entropy $S_1 = -\text{Tr} \rho_A \ln \rho_A$. These Renyi entropies characterize the pattern of entanglement in a quantum state, with higher entropy corresponding to more entanglement. This work examines the growth of these entropies (and their fluctuations) when a quantum system (which is initially low-entangled) is acted on by randomly chosen local unitary operations. This ‘random unitary circuit’ provides a minimal model for entanglement growth, in which no structure (apart from locality) is imposed on the time evolution operator. It thus provides a valuable yardstick against which any more structured model of time evolution could be compared, and remarkably, appears to yield universal results. In one dimension the dynamics of entanglement under random local unitary time evolution is argued to fall into the Kardar-Parisi-Zhang (KPZ) universality class of classical non-equilibrium dynamics, and a natural generalization is also proposed for higher dimensions.

The Kardar-Parisi-Zhang equation [4] is a nonlinear stochastic partial differential equa-

tion describing the time evolution of a scalar field $h(\mathbf{x}, t)$. It takes the form

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{x}, t) + c \quad (1)$$

where ν , λ and c are parameters and where η is Gaussian distributed white noise with mean zero, $\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2D \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$. Numerous problems in non-equilibrium classical statistical mechanics (in particular surface growth with $h(\mathbf{x}, t)$ being the height field), are known to be described by the KPZ equation, with the terms above being the ‘most relevant’ terms that are allowed to appear by symmetry. The authors of [1] propose that for a one dimensional system undergoing random unitary time evolution, the entanglement dynamics can be modeled by the above equation replacing $h(x, t)$ by $S_n(x, t)$ (any n), where $S_n(x)$ is the Renyi entropy of the subsystem extending from the left end to the point x .

The mapping to KPZ dynamics is remarkable for two reasons. Firstly, in the classical KPZ setting the terms appearing in the differential equation are fixed by symmetry. There is no such symmetry for quantum entanglement growth, and thus no reason why only the ‘KPZ’ terms should appear in the differential equation for entanglement growth. More profoundly, Renyi entropies are not local observables, and there is no reason why they should obey local Markovian dynamics. Nevertheless the authors are able to show that at least in two ‘solvable’ limits, the dynamics of the Renyi entropies is governed by the KPZ equation, and conjecture that this mapping should generically hold true.

This first model for which entanglement growth can be shown to fall into the KPZ universality class involves the application of random two spin quantum gates to a one dimensional spin q chain in the large q limit. In this limit, the authors show (by averaging over unitary two spin gates), that all of the Renyi entropies evolve according to the dynamical rule $S(x, t + 1) = \min(S(x - 1, t), S(x + 1, t)) + 1$, where x is the location of the bond where the unitary is being applied and t is the (discrete) time index. This is the same dynamical rule as for one dimensional surface growth, a problem that is described by the KPZ equation.

A second model where entanglement growth can be shown to fall into the KPZ class involves the time evolution of a spin half chain under application of a restricted set of random gates. A key step in this argument involves labelling a quantum state by its ‘stabilizer operators’ (those local operators for which the state is an eigenstate). The authors show that these ‘stabilizer’ operators grow ballistically in time under Heisenberg time evolution, much as is expected in generic thermalizing quantum systems [5]. The entanglement entropy

is then given by the number of stabilizer operators that span the cut - a number that grows linearly in time. *However* there is a ‘gauge freedom’ in the choice of stabilizer operators, such that the *speed* of entanglement growth is actually slower than the speed of growth of the stabilizer operators, because one can continuously make gauge transformations to a more local set of stabilizer operators. The detailed mapping to KPZ dynamics involves keeping track of the endpoints of the stabilizer operators, and mapping this motion to a biased random walk with interacting walkers, which is known to fall into the KPZ class.

Thus for two distinct (highly structured) models of quantum systems time evolving under the action of a random unitary circuit, the authors have shown that the dynamics of the Renyi entropies fall into the KPZ class. The authors then conjecture that this should be true for *generic* quantum systems under the action of a random unitary circuit. The intuition for this is based on a third picture - that of the ‘minimal length cut’ through the unitary circuit. The logic here is to ask what is the minimum number of legs of the local unitary circuit that one has to cut in order to prevent the system from getting entangled across that point (equivalently, how many local unitaries does one need to apply to remove the entanglement across a particular bond). This number should provide an upper bound on the entanglement in the system. The problem of finding the minimal cut then maps on to the problem of minimizing the energy of a directed polymer in a random medium, which is also known to fall into the KPZ class. This approach has the virtue of being model independent, and generalizable to higher dimensions (in two dimensions, one searches for the ‘minimal area cut’ instead). However, it only provides an *upper bound* on the entanglement, and the growth of entanglement could be slower than the growth of the minimal cut. Nevertheless the authors present numerical data on several ‘generic’ systems suggesting that, while free fermions violate this rule, entanglement dynamics under random unitary time evolution do generically seem to fall into the KPZ universality class.

A number of interesting questions are thrown open by this work. Firstly, how widely applicable is this result? Random unitary time evolution can be viewed as a reasonable model for fully coherent Hamiltonian dynamics in the presence of *classical* noise (as the authors note), but should we expect similar behavior for quantum dynamics coupled to a chaotic (i.e. thermalizing) *quantum* system? In this case there is the possibility of entanglement between the two coupled quantum systems, which may substantially impact the entanglement dynamics. Equivalently, does random unitary time evolution provide a good model

for entanglement dynamics in the presence of classical noise where we allow the noise to decohere the system i.e. Lindblad evolution? Or are the results specific to closed quantum systems? If they are specific to closed quantum systems, do they apply to Hamiltonian dynamics without noise? Even if noiseless Hamiltonian evolution does not fall into the KPZ class, can the ways of thinking introduced in this paper (like the ‘directed polymer’ picture) be usefully applied? Do the broad conclusions (such as entanglement spreading being slower than operator growth) carry through? Under what circumstances might the framework introduced for random unitary circuits be a good model for noiseless Hamiltonian time evolution? For instance - is ergodicity required? Is it sufficient?

A better understanding of these questions would clarify what physical systems might be described by this model. Other interesting questions include whether there is a meaningful semiclassical limit (i.e. where does \hbar appear), whether there is any dependence on Renyi index in generic models (there is no such dependence in the special models studied here), and whether there is any connection to earlier work on ‘slowest growing operators’ [6]. Finally, the authors proposed generalization to higher dimensions, whereby the growth of entanglement is bounded by the ‘minimal area cut’ through the quantum circuit, is reminiscent of holographic approaches to the calculation of entanglement entropy [7]. Is the similarity superficial, or is there a deep connection to holography?

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