Quantum Entanglement sheds new light on an old problem

A Proof of Deconfinement in a 2D Gauge Theory from Quantum Entanglement

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Recommended with a commentary by Ashvin Vishwanath, UC Berkeley

In his seminal paper on the renormalization group (RG), Ken Wilson likens the RG flow of coupling constants to a ball rolling down a hill. In this analogy, the equilibrium positions of the ball, the RG fixed points, correspond to phases of matter and phase transitions. A natural question that arises on pushing this analogy is - can we assign an 'altitude' to fixed points? If perturbations nudge a system away from a fixed point, we could immediately conclude that the only allowed final states are the ones at a lower 'altitude' (see figure). While in general RG flows are more complex than this picture suggests, a certain class of gapless one-dimensional quantum systems (or two dimensional statistical mechanics models) do indeed play out this scenario. For states with conformal symmetry (such as the critical point of the 2D Ising model) the 'c-theorem' assigns a real number $c \geq 0$ which can be interpreted as an altitude. Perturbations can only decrease c - which provides strong constraints on phase diagrams. Subsequently, analogous theorems have been proposed to apply to quantum systems in spatial dimensions D=2 (the f theorem) and D=3 (the a-theorem). Again, conformal symmetry is an important ingredient, but this is naturally realized in several cases such as in the 2+1D critical point of the quantum Ising model (or equivalently in the 3D Ising model at its finite temperature transition) .

The highlighted paper by Grover makes clever use of the f-theorem to argue that a model of Dirac fermions interacting via gauge forces (Quantum Electrodynamics in 2+1D or QED_3) must have a deconfined phase. That is, unlike the confining effect of gauge forces that bind quarks into mesons, hadrons etc., the fermions persist to low energies in this model, forming a strongly coupled soup of excitations.

The action of the model itself, QED3, is

$$\mathcal{S} = \int d^2x\, dt \sum_{j=1}^{N_f} \bar{\psi}_j \gamma \cdot (p - q\, A)\psi_j + (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$

It has N_f 'flavors' of 4 component fermions (in graphene where Dirac fermions also arise, $N_f = 2$ corresponding to the two spin projections). However, unlike graphene, the gauge

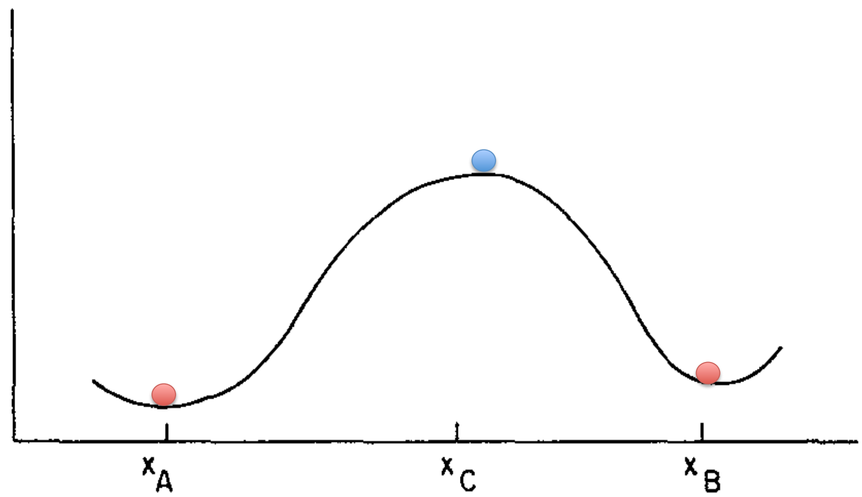

FIG. 1: The fixed points x_a and x_b can be reached by perturbing x_c. Hence we expect f_{x_a}, $f_{x_b} < f_{x_c}$ for 2+1D conformal theories.

potential A is emergent and constrained to propagate solely in two spatial dimensions leading to strong interactions between fermions. Previously QED_3 has been invoked to describe a variety of phenomena from the pseudogap state of high Tc to the ground state of frustrated quantum antiferromagnets on the Kagome lattice. However, the basic question of its stability has been debated. When the gauge coupling is turned on - i.e. by introducing $q > 0$, it is found to grow under RG. The question is, does this growth stops at some finite charge, leading to a strongly interacting fixed point, or continue to grow till the gaplessness itself is destroyed? It is known that with increasing number of flavors of fermions N_f, the theory is more likely to be stable. However what this minimum number is was only being addressed via complex numerical calculations or large N_f techniques [1]. Grover has found a completely different approach to argue for stability exploiting, ultimately, basic properties of quantum entanglement.

This is where the f-theorem enters. Let us take a detour to discuss this. The quantity f is most physically defined in terms of quantum entanglement. Define a circular region 'A' with perimeter L and calculate the entanglement entropy S_A associated with this region by the usual von Neumann formula:

$$
\begin{aligned}
S_A &= -\mathrm{Trace}_A \left[\rho_A \log \rho_A\right] \text{ where} \\
\rho_A &= \mathrm{Trace}_B |\Psi\rangle\langle\Psi|
\end{aligned}
$$

Then the entanglement entropy (in a 2+1D conformal theory) is expected to take the form:

$$S_A = \alpha L - f + \ldots$$

that is, in addition to the leading term that grows with the circumference, there is a subleading constant which is f. Following a number of explicit calculations of f it was verified that f plays a role akin to the central charge in 1+1D CFTs. If perturbing a fixed point with f_1 destabilized it leading to another fixed point with f_2, then $f_1 > f_2$. A proof of the f theorem was offered in Ref.[2] given below that makes central use of a fundamental property of quantum information, that we will return to at the end of the article.

Grover makes use of this theorem to argue that for sufficiently large N_f, QED_3 is stable. A crude sketch of the proof goes as follows. One likely instability is chiral symmetry breaking, where fermions acquire a mass gap once an expectation value $\langle\bar{\psi}_i\psi_j\rangle \neq 0$ develops. This however breaks flavor symmetry and leads to $2N_f^2$ Goldstone modes. Thus while the contribution to f from the individual fermion modes scales linearly $f_1 \sim 2N_f\gamma_1$, the contribution of Goldstone modes in the symmetry broken state scales as the square of the number of flavors: $f_2 \sim 2N_f^2\gamma_2$. If $f_2 > f_1$, chiral symmetry breaking will be ruled out as an instability, which, putting in the numbers yields $N_f > N_f^c = 3.3$. That is, beyond a certain critical number of flavors, QED_3 is stable. The argument above has glossed over various subtleties, such as the contribution of the photon to f which is formally divergent, and the case where monopole excitations are present where other instabilities can intervene. These issues are addressed in the paper, by invoking supersymmetry and a theorem of Vafa and Witten - but the net upshot is that while N_c is raised, it remains finite. Thus the qualitative result that QED_3 is stable with a sufficiently large number of fermion flavors continues to hold.

This result ties together two very different worlds. A key ingredient in Ref [2] in proving the f-theorem is that quantum information satisfies the following property termed strong sub-additivity. For a pair of overlapping regions A and B, the entanglement entropy satisfies:

$$S(A \cap B) + S(A \cup B) \leq S(A) + S(B)$$

It is remarkable that this fundamental property of quantum entanglement ultimately constrains the possible dynamics of a strongly interacting field theory! Many of the results mentioned here will need further work to set them on a rigorous footing. But already in this example, quantum entanglement ideas added fresh life to an issue that has stagnated for decades.

[1] On the stability of U(1) spin liquids in two dimensions M. Hermele et al., Phys. Rev. B 70, 214437 (2004).

[2] On the RG running of the entanglement entropy of a circle, H. Casini, Marina Huerta, arXiv:1202.5650.