Majorana fermions and half-quantum vortices in superfluid 3-He

Observation of Half-Quantum Vortices in Topological Superfluid 3-He
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One of the most fascinating ideas to emerge in condensed matter physics over the last two decades has been the concept of topologically protected quantum computing (hereafter abbreviated TQC)-that is, crudely speaking, the idea that one may be able to protect the information required to carry out a quantum computation from the effects of decoherence by the environment, and manipulate it in a robust manner, by burying it in the complicated degrees of freedom of a strongly interacting many-body system in a way which makes it proof against any kind of local probe[1,2]. The simplest implementations of TQC require a system with 2 (and no more than 2) spatial dimensions, which is why in the following I shall often specialize to this case.

One particularly attractive proposal for realizing TQC has involved the use of Majorana fermions. As the name implies, this exotic type of excitation was originally proposed in the context of particle physics by the Italian theorist Ettore Majorana in one of his last papers[3]; it is a fermion which is its own antiparticle (a state of affairs not unusual for bosons, but much less well-known in the fermionic sector). As of this writing, there is no firm evidence for the existence of such particles in high-energy physics. However, it has been pointed out that in many-body physics, at least to the extent that one accepts the standard analogy between the antiparticles of Dirac's original electron theory and holes in a Fermi sea, an exactly parallel state of affairs is possible. While there are by now proposals to realize it (and in some cases, claims to have done so) in several different physical systems, the one which is directly relevant in the present context is a so-called "$p+ip$" bulk Fermi superfluid, and I will confine myself here to that system.

A "$p+ip$" Fermi superfluid is a system of fermions, usually with two spin species, in which Cooper pairs form, similarly to a BCS superconductor, in a linear combination of states $\boldsymbol{p},-\boldsymbol{p}$ but with an order parameter (pair wave function) whose dependence, for $\boldsymbol{p}$ close to the Fermi surface, on the direction of the latter is p_x+ip_y (hence the name). Moreover, since this expression is an odd function of p, to preserve the correct fermionic antisymmetry the spins of the particles forming the pair must be parallel; while more complicated cases are possible, in most systems of current interest the pairing is simply of an "up" with an "up" and a "down" with a "down" ("equal-spin" pairing, ESP); in this case, for many purposes the up and down spins may be thought of to a first approximation as independent systems. The currently known physical systems which are thought to realize the ESP $p+ip$ state are, with some confidence, the A phase of superfluid 3-He and, with somewhat less confidence, the superconductor strontium ruthenate (Sr_2RuO_4) and possibly one or two heavy-fermion superconductors; in future it may be possible to realize this state also in some ultracold alkali gases. An interesting property of this state is that while in three spatial dimensions the energy gap has nodes in the p_z direction, if the system is confined to a plane the gap, which is proportional to the magnitude of the pair wave function, is a nonzero constant everywhere over the Fermi "ring".

Let us now consider the nature of the simplest fermionic excitations (Bogoliubov quasiparticles) of the system. The standard method of obtaining these excitations, which has long been applied to simple "BCS" superconductors in which the pair wave function is of the singlet s-wave form and has more recently been generalized to more exotic types of pairing, is to make a so-called mean field approximation ($\widehat{H} \to \widehat{H}_{mf}$) for the Hamiltonian (thereby reducing the law of conservation of particle number to conservation *mod* 2) and write the creation operator for the quasiparticle in the form

$$\alpha_i^\dagger = \int d\boldsymbol{r} \left\{ u_i(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}) + v_i(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) \right\} \tag{1}$$

where $\psi^\dagger(\psi)$ is the creation (annihilation) operator of a real fermion (electron or 3-*He* atom, as the case may be); thus $u(\boldsymbol{r})$ is the amplitude for creation of an extra real fermion and $v(\boldsymbol{r})$ that for subtraction of one (or equivalently for creation of a hole). By demanding that the operation of $\alpha_i^\dagger$ on the (even-*N*) groundstate produces an odd-*N* energy eigenstate with energy E_i (relative to the chemical potential), *i.e.* formally satisfies the relation

$$\left[\widehat{H}_{mf}, \alpha_i^\dagger\right] = E_i a_i^\dagger \tag{2}$$

one obtains a pair of equations for $u_i(\boldsymbol{r})$ and $v_i(\boldsymbol{r})$ which involve the pair wave function (order parameter); these equations are known as the Bogoliubov-de Gennes (BdG) equations, and have formed the basis of almost all discussions of inhomogeneous superconductivity in the theoretical literature for the last 50 years. In the case of an ESP superfluid the BdG equations for the up- and down-spin particles decouple to a first approximation (which is why I have not written a spin suffix explicitly on the psi's).

Let's now specialize to the (ESP) $p+ip$ case. If one considers a simple "bulk" situation in which the order parameter may have substantial spatial inhomogeneity but has no topological singularities, the result of solving the BdG equations is not particularly interesting; one finds, except possibly near the edges of the system, a fermionic spectrum qualitatively similar to the well-known form for the spatially uniform case, $E_k = \sqrt{\varepsilon_k^2 + |\Delta_k|^2}$; in particular, in the 2D case there are no excitations with energy much less than the uniform energy gap δ. However, things become much more interesting when one considers a vortex, that is a line (in 2D point) around which the phase of the order parameter (or more precisely, the dependence of the latter on the center of mass of the pair) circulates through 2π. Let's for simplicity specialize to 2D (or in 3D, to solutions independent of z). It turns out that the BdG equations now allow, in addition to various other low-(but nonzero-) energy solutions qualitatively similar to those which occur in the s-wave case, a single special solution with the properties:

(1) the amplitudes $u(\boldsymbol{r})$ and $v(\boldsymbol{r})$ are localized close to the vortex.

(2) $u(\boldsymbol{r}) = \boldsymbol{v}^*(\boldsymbol{r})$

(3) $E = 0$

(actually, (2) and (3) are not independent). This is the famous Majorana fermion; as can be seen from property (2), the "particle" it describes is its own antiparticle. What makes this exotic excitation specially interesting in the context of TQC is two properties:

(A) Because by (2) it is an exactly equal mixture of particle and hole, its presence does not change the local particle density (or any other local single-particle quantity) and thus it is completely undetectable by any local probe.

(B) Because as we will see below it is really only "half" of a genuine fermionic excitation and may be widely separated from the other half, its behavior under braiding is very robust and nontrivial; in particular, such braiding can realize a subset of quantum-computing operations in a topologically protected way.

What exactly is a Majorana fermion? I believe that much of the existing literature on this question has been unnecessarily confused by reference to the alleged "negative-energy" solutions to the BdG equations. Let us forget these, and rather proceed as follows[4]: when applied to the even-*N* groundstate, a solution to eqn. (2) with eigenvalue zero has two possible interpretations: The obvious one is that $\alpha^\dagger$creates an odd-*N* state which is energetically degenerate with the even-N groundstate ("zero-energy Bogoiluibov quasiparticle"). However, an equally valid interpretation is that $\alpha^\dagger$ simply annihilates the even-*N* groundstate, *i.e.* when acting on it produces a vector of zero norm! Now it can be readily shown[4] that neither of these interpretations by itself allows the self-conjugacy condition (2) to be satisfied; however, by forming a quantum superposition of the two it is possible to satisfy it. Thus we reach the conclusion that _a Majorana fermion is simply a quantum superposition of a true zero-energy Bogoliubov fermion and a "pure annihilator"! In the literature it is more common to turn this statement around and note that by combining two Majoranas with appropriate relative phase it is possible to produce a true zero-energy Bogoliubov fermion. This of course requires that the number of MF's for any given physical $p+ip$ Fermi superfluid is always even, and this indeed turns out to be the case (if the number of vortices is odd, an extra solution appears on the system boundary). However, the striking point is that the two "halves" of the Bogoliubov fermion-the two MF's-can be arbitrarily distant in space (and moreover, in the case of 2n MF's with $n > 1$, their association into n Bogoliubov fermions is not unique); hence the topological protection discussed above. (In the present authors's opinion, it is much clearer to discuss TQC in $p+ip$ superfluids in the language of real Bogoliubov fermions than in that of MF's).

Now, there is plenty of evidence for the occurrence of vortices both in 3-*He-A* and in strontium ruthenate. Moreover, the strongly layered structure of the latter makes it plausible that to a first approximation each individual plane may be treated as a "2D" system, and in the former case it may be possible with improved cryogenics to obtain an "effectively 2D" situation (cf. below). Thus, at first sight it would seem that everything is in place to implement TQC in one or both of these systems. This is an exciting prospect which has generated a large theoretical literature over the past 15 years.

Alas, there is a snag: The vortices which are routinely seen in these two ESP systems are "standard" vortices in which the up- and down-spin fermions circulate together. Even if MF's indeed occur in these vortices, there will be two of them (one for each spin population) occurring for any given vortex, and then, unless they have literally zero contact (a pathologically improbable case), it will be possible to combine them to make a localized Bogoliubov fermion, which is not expected to have any particularly exotic properties or to permit its use for TQC. So it is necessary to consider so-called "half-quantum" vortices (HQV), in which, crudely speaking, the up spins circulate as in an ordinary vortex while the down spins have no singularity (in a charged system these vortices trap $h/4e$ of flux, hence the name); each such vortex will indeed host one and only one M.F., and the proposals for TQC can go forward.

These HQV's are of interest in their own right, and searches have been carried out for them in both 3-*He*-*A* and strontium ruthenate. In the latter case Jang et al.[5] produced evidence from magnetotorque measurements of $h/4e$ flux quantization in a ring geometry, but to date there has been no observation of even a single HQV in bulk, let alone the large number which would be necessary for meaningful TQC. In superfluid 3-*He*, searches in the bulk *A* phase have until now yielded negative results. However, in the Letter[6] commented on Autti et al. present NMR evidence for their occurrence (in substantial numbers) in a different phase, the so-called "polar" phase which has a real order parameter proportional to p_x (or p_y) and which has been identified as occurring in certain kinds of restricted geometry (in this case, in the pores of the nanomaterial nafen). The HQV's can be produced by rotation of the container while cooling into the superfluid phase, but also occur under zero rotation, perhaps by the so-called Kibble-Zurek mechanism; the production rates seem reasonably consistent with theoretical expectations.

The polar phase of superfluid 3-He has a real order parameter and thus presumably does not sustain M.F's in its vortices. However, in other types of restricted geometry a related phase, the so-called polar-distorted *A* phase, has been detected[7]. This has an order parameter of the form p_x+ikp_y with $0 < k < 1$; if HQV's can be produced also in this system, they should sustain MF's and (provided that the latter are not too strongly pinned to the substrate, as appears to be the case in the experiments of ref. [6]) in principle permit TQC. Of course there are any number of other practical difficulties which will need to be overcome (e.g. making a slab of 3-*He* which is thin enough to count as "2D" at an achievable temperature while still remaining superfluid), but the experiment of ref. [6] is at least a valuable step in that direction.

References:

1. A. Yu. Kitaev, Phys. Uspekhi 44, 131 (2001)
2. C. Nayak et al., Revs. Mod. Phys. 80, 1083 (2008)
3. E. Majorana, Il Nuovo Cimento 14, 171 (1937)
4. A. J. Leggett, in Doing Physics: a Festschrift for Tom Erber, IIT Press, Chicago 2011, p.173
5. J. Jang et al., Science 331,186 (2011)
6. S. Autti et al., PRL 117, 255301 (2016)
7. R. Sh. Askhadullin et al., JETP Letters 100, 662 (2014)