Height variables in the Abelian sandpile model:scaling fields and correlations
Authors: Monwhea Jeng, Geoffry Piroux and Philippe Ruelle
arXiv.org/cond-mat/0609284
J. Stat. Mech.(2006) P10015

Recommended with a commentary by
D. Dhar, Tata Institute of Fundamental Research, Mumbai

The Abelian sandpile model was introduced by Bak, Tang and Wiesenfeld in 1987, as a prototypical model to illustrate the basic concepts of self-organized criticality. The model has generated as a lot of interest, and many variations of the model have been studied over the last two decades.

The lasting interest in the model comes from the fact that it is one of the simplest models showing self-organized criticality, it is easy to define, and seems to be exactly soluble. It has a an interesting Abelian group structure, which allows the exact calculation of several properties of the critical steady state, without too much trouble. For example, one can exactly characterize the probability of occurence of different configurations in the steady state, and determine the spectrum of relaxation times fairly easily, using the abelian group structure of the model, in all dimensions.

Other properties of the model are not so easy to determine. In particular, a major focus of interest has been the determination of critical exponents that characterize the power-law tail of the distribution of the size of avalanches in the model, and despite a lot of theoretical and numerical effort directed towards this end in the last twenty years, these are still not known exactly, or with good precision numerically, for the original BTW model defined on the square lattice.

The growth and decay of avalanches depends, in a complicated way, on the correlations in the heights at different sites in the critical steady state. The latter are quantified by the different n-point correlation functions of heights. The simplest of the correlation functions is the 1-point correlation function. which is specified by $P(j)$, the probabilities that a randomly picked site will have height j [$j = 1$ to 4, for the abelian sandpile model on the square lattice].

The calculation of $P(1)$ was first done by Majumdar and Dhar [1]. It uses the one-to-one correspondence between recurrent configurations of the sandpile, and spanning trees. Priezzhev [2] extended this technique to determine $P(2), P(3)$ and $P(4)$. However, these calculations are rather complicated, and Priezzhev's paper only gave details of calculation of $P(2)$. This is expressed as a lattice sum over all $\vec{R}$ of a determinant of a 4×4 matrix whose elements are the lattice propagators of the type $G(\vec{R} + \vec{e}_\alpha)$, where $G(\vec{R})$ is the lattice propagator at separation $\vec{R}$ [Its Fourier transform is $1/(2 - cosk_x - cosk_y)$], and $\vec{e}_\alpha$ are lattice vectors.

Priezzhev did not succeed in evaluating this sum in a closed form, though he did simplify the result a bit more, with the final expression being a single multidimensional integral that had to be evaluated numerically. It seemed hard to push the technique much further, and that is where the matters stood for a while.

Jeng et al follow Priezzhev's general technique of graphical enumeration of classes of graphs using the equivalence to spanning trees, but they are able to guess the exact values of P(j) from their formulas. They find that

$$P(1) = \frac{2}{\pi^2} - \frac{4}{\pi^3} \tag{1}$$

$$P(2) = \frac{1}{4} - \frac{1}{2\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3} \tag{2}$$

$$P(3) = \frac{3}{8} + \frac{1}{\pi} - \frac{12}{\pi^3} \tag{3}$$

$$P(4) = 1 - P(1) - P(2) - P(3) \tag{4}$$

These formulas for $P(j)$'s are stated as conjectures by Jeng et al, as the expressions they find for these involve a definite integral J_2 of the type $\int_0^\pi d\theta \int_0^\pi d\phi F(\theta, \phi)$, with the integrand F being a rather complicated function of sines and cosines of linear combinations of θ and ϕ. This integral has been evaluated by them numerically, and has a value 0.5 to some twelve digit accuracy. If one can prove that the integral is *exactly* $1/2$, then the conjectured results follow.

The existence of these simple formulas for the height probabilities suggests that there might also be a simpler derivation, which will constitute a major advance towards answering other outstanding questions related to this model.

Jeng et al go beyond the calculation of 1-point functions. They have also calculated 2-point and higher correlation functions of heights. One can define a field variable

$$h_i(\vec{x}) = \delta(z_{\vec{x}}, i) - P(i). \tag{5}$$

where δ is the Kronecker δ-function. The expectation value of this field is zero by construction, and multipoint correlation functions give the multi-site joint probabilities of different heights. It was known that the field $h_1(\vec{x})$ has correlations $\langle h_1(\vec{x}) h_1(\vec{x'}) \rangle \sim |\vec{x} - \vec{x'}|^{-4}$. Jeng et al calculate explicitly the correlation functions of h_i, and find that in general

$$\langle h_i(\vec{x}) h_j(\vec{x'}) \rangle \simeq \frac{1}{R^4} [a_i a_j \log^2 R + C(a_i + a_j) \log R + D_{ij}] \tag{6}$$

where $R = |\vec{x} - \vec{x'}|$, and explicit values of the constants a_i and C and D_{ij} have been determined. It thus appears, that in the continuum limit, all the four fields $\{h_i\}$ with $i = 1$ to 4 can be described as linear combinations of only two fields $\phi(\vec{x})$ and $\psi(\vec{x})$, with

$$h_i(\vec{x}) = \alpha_i \phi(\vec{x}) + \beta_i \psi(\vec{x}), \tag{7}$$

where $\{\alpha_i, \beta_i\}$ are related to the constants $\{a_i, D_{ij}\}$ that appeared in Eq.(6), and $\phi(\vec{x})$ is a field of in a conformal field theory with central charge -2, and the $\psi(\vec{x})$ field is the "logarithmic partner" of $\phi(\vec{x})$. In the logarithmic conformal field theory [3], the correlation function $\langle \phi(\vec{x})\phi(\vec{x'})\rangle$ varies as simple power law $|\vec{x} - \vec{x'}|^{-4}$, but correlation functions involving ψ field also have finite degree polynomials of $\log |\vec{x} - \vec{x'}|$ multiplying the power law decay.

The powerful techniques of conformal field theory constrain strongly, and almost determine fully, the functional forms of the n-point correlation functions of the sandpile. Moreover, as the sandpile model (equivalently, spanning trees) provides a simple statistical mechanical realization of the rather abstract theoretical structure of the logarithmic conformal field theory, it is reasonable to expect that further study of this connection will help understand better the latter as well.

References

[1] S. N. Majumdar and D. Dhar, J. Phys. A: Math. Gen., **24**, L357 (1991).

[2] V. B. Priezzhev, J. Stat. Phys., **74**, 955 (1994).

[3] V. Gurarie, Nucl. Phys. B **410**, 535 (1993).