Geometric origin of low-frequency excess density of vibrational modes in: weakly-connected amorphous solids

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Elastic solids are stable against small deformations. The energy of these deformations is a positive, quadratic function of the displacements which however is strictly zero for uniform displacements. Therefore, quasi-uniform displacements, i.e. sound waves with large wavelengths λ, cost a small energy $\sim \lambda^{-2}$. From the dispersion relation $\omega = 2\pi c/\lambda$, where c is the speed of sound, and the usual counting of modes in wavevector space, one easily finds that density of vibration modes $g(\omega)$ is given by the Debye law $g(\omega) \sim \omega^2$.

There are many situations where the "solid" under consideration is in fact only barely stable. For example, if one compresses a gas of (quasi-)hard spheres at zero temperature, there will be a density where the system jams and some non zero elastic moduli appear. Right at the jamming transition, the number of contacts is just sufficient to allow the system to be mechanically stable, at situation called isostatic. Similarly, when a liquid is hyperquenched to very low temperatures, it falls in locally stable configurations that are very far from thermal equilibrium, but at least mechanically stable. More stable configurations cannot be reached because barrier crossing is forbidden at these low temperatures. In these situations, the relevant configurations are not the low energy ones, but the first locally stable configurations reached by the dynamics. The importance of these "marginal" states has been pointed out by Shlomo Alexander, Sam Edwards and many others, in particular in the context of granular media. Similar ideas appear in the context of Coulomb glasses and the Efros-Shklovskii gap [1] and of spin-glasses. For example, one can show analytically that in some mean-field spin-glass models, the dynamics at low temperature dynamics drives the system to marginal states that can be viewed in energy space either as minima with zero cost directions, or saddle points with a vanishing number of unstable directions [2].

Therefore, quite generically, one expects these marginal states to have a large number of low energy, "soft" modes, vestiges of the unstable directions that just became stable as the density increased, or as the energy decreased. The argument developed by M. Wyart et al. shows that for an elastic solid with a marginal number of contacts, the density of low-frequency modes is in fact *constant*, $g(\omega) \sim \omega^0$, and therefore much larger than the Debye density in a usual elastic solid. Their argument is variational: they construct trial deformation modes with a bounded energy to obtain a lower bound on the density of states. Physically, the reason for this huge number of modes is as follows: if one removed all L^2 contacts on a plane $\mathcal{P}$ cutting a system of size L, L^2 unstable modes would appear (since the original configuration had just enough contacts to be stable). Each one of these modes can be modulated in space such as to have zero displacement on the plane $\mathcal{P}$ and approximate a true low energy mode of the original (uncut) configuration. The modulation of wavelength $\lambda = L$ is easily shown to increase the energy of the phantom zero mode by a quantity $\sim 1/L^2$, as in an elastic body. However, at variance with an elastic body, there are now $\sim L^2$ modes of frequency $\sim 1/L$, consistent with a constant density of states at low frequencies. [Using a real-space renormalisation group like argument, this result can then be extended up to frequencies of order 1, not only $1/L$.] As the number of contacts is increased from criticality, a characteristic length scale appears ℓ^*; the system is marginal (isostatic) for smaller length scales, and stable at larger length scales. Therefore, a plateau in the density of states is expected for $\omega > \omega^* = c/\ell^*$.

This finding is important because extremely generic; it might in particular explain the ubiquitous "Boson-peak" in glasses, which is an increased density of vibrational states around a frequency in the THz region. Recent work by the Parisi group [3] have indeed suggested that this increased density of states is related to the proximity, in energy space, of a critical point where unstable saddle points become stable minima, as predicted by the Mode-Coupling theory of glasses. The geometrical, intuitive approach of M. Wyart et al. sheds a very interesting light on this problem, and might allow one to understand the role of pressure, cooling rate and aging effects on the height and frequency of the Boson-peak. It also points out the existence of an important crossover length in glasses and granular materials, below which the mechanical properties of the structure is very far from that of a usual elastic body. This could be important to understand, for example, the fracture properties of glasses [4].

[1] For a recent interesting piece of work, see M. Mueller, L. B. Ioffe, *The glass transition and the Coulomb gap in electron glasses*, cond-mat/0406324.

[2] J. Kurchan, L. Laloux, J. Phys. A, **29**, 1929 (1996).

[3] T. S. Grigera et al., Nature, **422** 289 (2003).

[4] F. Célarié et al. Phys. Rev. Lett. **90** 075504 (2003).