Of the knot but not the knot

Isolated optical vortex knots
Nature Physics **6** (2009) 118.
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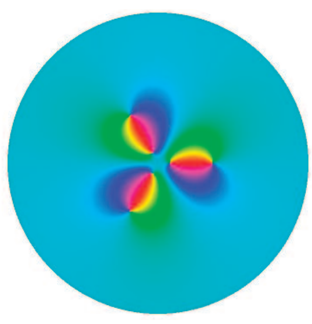
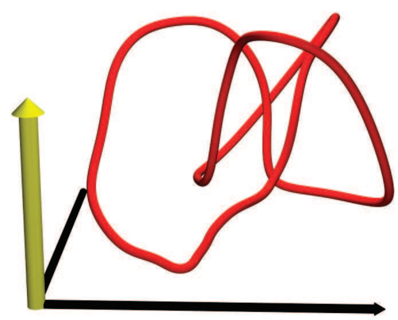
Recommended and a commentary by Randall D. Kamien, University of Pennsylvania

Few things illustrate the beauty and mystery of topology as knots in three dimensions. The Alexander and Jones polynomials are, for instance, convenient tools to determine if two knots are different, but determining if two knots are the same is notoriously difficult. Moreover, all knotted loops can be mapped smoothly to each other as all of them are homeomorphic to the circle. What distinguishes two knots then is the space that is *left over* when the knot is removed from three-space $\mathbb{R}^3$ (or, to be precise, the three-sphere, $\mathbb{S}^3$). Suppose now that this knot is somehow entrained or attached to a field in the three-dimensional space, possibly realizing Kelvin's vision of knots in the æther making atoms or, less fancifully, realized by a complicated (let's reserve complex for *complex plane*) velocity field in a fluid. Indeed, were it not for viscosity, smoke rings and the flow around them would be stable knots with the associated knotted field. Recall that a simple closed loop is called the "unknot" by topologists so one might say that smoke rings and superfluid vortex loops generate "unknotted fields."

Since each polarization of an electromagnetic field moving along the z-axis is a complex scalar, it follows that nodal lines of the field (or "zero set") form at the center of phase vortices. With this insight, the authors have made a major step forward in the control of light. First, using a known mathematical prescription, the Milnor map, they construct a polynomial in two complex variables which encodes the desired knot as its zero set. By stereographic projection this can be turned into a complex field in space. So far so good, but this is not a new construction. What the authors have discovered is that if the phase field on the xy-plane is given as initial conditions to *the paraxial equation* (the Schrödinger equation with the z-direction replacing time),

$$-i\partial_z\psi = \left(\partial_x^2 + \partial_y^2\right)\psi$$

then the evolved phase field sometimes encodes precisely the same knot. Though there is no mathematical understanding of why this should be true, the proof of the knotting is in the beating of the superposition of Laguerre-Gaussian beams. The left-hand figure, shows the

$z = 0$ section of the complex phase field (color indicates phase – see paper for colorscale). The right-hand figure shows the measured (not computed!) knot, the trefoil.

The authors' success raises a number of mathematical questions – which knots can be constructed this way? Do other evolution equations share this amazing feature? Replacing z with time, is there any application for the knots in time solutions of the Schrödinger equation? One might also speculate on applications of these knotted waves from communication to computation. Like the knotted "khipu" strings of the Incans [2], will it be possible to encode information in the twists and turns of the nodal sets of light? More generally, can knots be tied into the topological defects of superconductors, superfluids, and liquid crystals? Can these 21$^{\text{st}}$ century atoms be used as building blocks for new materials?

- [1] K. O'Holleran, M.R. Dennis, and M.J. Padgett, *Topology of light's darkness*, Phys. Rev. Lett. **102**, 143902 (2009).
- [2] C.C. Mann, *Cracking the Khipu Code*, Science **300**, 1650 (2003); *Unraveling Khipu's Secrets*, Science **309**, 1008 (2005).