Geometric frustration in twisted strips

J. Chopin and A. Kudrolli, "Helicoids, Wrinkles, and Loops in Twisted Ribbons," Phys. Rev. Lett. **111**, 174302 (2013).

Recommended with commentary by Chris Santangelo, UMass Amherst

Take a strip of paper, perhaps under gentle tension, and twist it in your hands. Does it deform gently, wrinkle, crumple, or tear? Were the sheet to helically twist it would necessarily also stretch, since the outside edge would be longer than the central axis [1]. Yet stretching is energetically unfavorable compared to bending when the sheet is very thin. The large separation of scales arising within the bending and stretching energies indicates that in-plane stresses should be "expelled" at the cost of bending. On the other hand, twisting a strip without stretching it introduces singularities at which the bending energy would diverge [2, 3]. Typically, these singularities are virtual – beyond the boundary of the sheet – but the shape becomes more highly curved as they are brought toward the boundary. So does an elastic sheet retain the singularities of the inextensible limit or does it instead wrinkle to balance stretching and bending energy more equitably? Despite the relative antiquity of the theory of elasticity, there does not yet seem to be a general agreement on how the tension (pun absolutely intended) between stretching and bending is resolved.

One part of the difficulty with this problem lies with the fact that the bending energy is singular perturbation, thus an elastic sheet can exhibit sharp, stress-focussed structures which devolve into singularities as the thickness vanishes without prohibitive energetic cost. The other difficulty rests with the fact that elasticity is geometrically nonlinear – the strain is quadratic in the displacement – making it difficult to determine how best to relieve elastic frustration. Indeed, often sheets will exhibit both [4].

One approach to make further headway in this question is to identify experimental systems that display multiple morphologies with a small number of easily controlled parameters. This is precisely what Julien Chopin and Arshad Kudrolli have done. They consider the equilibrium shape of a narrow strip twisted by a rotation angle Θ and under some tension T and identify several regimes: helicoid, longitudinal buckling, transverse buckling, creased helicoid, and loop (in their nomenclature). A key finding is that there is a critical point

(Θ^*, T^*). For $T > T^*$, they see a uniformly-twisted helicoid below a critical twist angle. At sufficiently high twist angle, however, the helicoid is unstable to transverse buckling and, eventually, the strip enters a messy self-contact regime.

For $T < T^*$, the morphology diagram looks quite different, however. As the twist angle rises, the helicoid gives way to longitudinal wrinkles. As the twist angle is increased, the longitudinal wrinkles continuously deform to a creased helicoid structure, which twists but does so along sharp ridges reminiscent of the classical origami fold pattern, the "wind spinner." Like its folded cousin, the creased helicoid exhibits high curvature ridges spanning across its width, and can be modeled by low energy configurations which, nevertheless, exhibit no stretching [2]. Finally, at large enough twist, there is a mysterious second transition to a loop state in which the twisting itself is confined to a localized region of the strip. This transition is hysteretic and poorly understood. The authors' scaling analysis, and subsequent experimental comparisons, indicates that the longitudinal and transverse buckling instabilities meet at the critical point, (Θ^*, T^*). Experiments seem to hint that the discontinuous transition from creased helicoid to loop also meets at this critical point.

Above all, this paper is a challenge to theorists. Here, we have an experimental system that exhibits a wealth of morphological behavior as a function of a few parameters. Is there anything that can be said beyond the linear stability analysis of a uniform state? How does a smooth, wrinkled state become sharply creased? These are questions that have been asked before, but maybe now there is a possibly to answer them – at least in one system.

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