Is there a speed limit for thermalization?

• *S. Sachdev and J. Ye, "Gapless spin-fluid ground state in a random quantum Heisenberg magnet," PRL 70 3339 (1993), cond-mat/9212030*

• *J. Maldacena, S. H. Shenker, and D. Stanford, "A bound on chaos," JHEP 8 (2016) 106, hep-th/1503.01409*

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The purpose of this note is to describe how a model [1] of interacting spins is related to recent ideas in quantum gravity and the foundations of quantum statistical mechanics. These surprising connections underscore how ideas in seemingly disparate areas of physics can influence and inform one another. Our intention is not to motivate the spin model through the experimentally-accessible systems it is intended to describe – e.g., see the excellent discussion in [2] of its possible relevance to heavy-fermion superconductors – rather, we will concentrate on the line of thought that connects a few underlying theoretical concepts.

The basic tenet of quantum statistical mechanics states that, at long times, a (sub)system will equilibrate with its environment, i.e., thermalize. More precisely, a quantum state $|\Psi\rangle$ of the total system is thermalized if for any sufficiently small subsystem S, the reduced density matrix $\rho_S = \mathrm{tr}_E|\Psi\rangle\langle\Psi|$ is *approximately* thermal. The trace is taken over the complementary degrees of freedom describing the environment E. Thus, at sufficiently long times t[1], expectation values of any operator $\mathcal{O}$ in S will have equilibrated,

$$\langle\mathcal{O}(t)\rangle \equiv \mathrm{tr}_S\Big(\mathcal{O}(t)\rho_S(t)\Big) \rightarrow \frac{\mathrm{tr}_s\Big(\mathcal{O}(0)e^{-\beta H_S}\Big)}{\mathcal{Z}}, \tag{0.1}$$

where the trace is over eigenstates, labeled by s, of the operator H_S, $\mathcal{Z} = \mathrm{tr}_s e^{-\beta H_S}$, and $\rightarrow$ denotes an *appropriate* long-time limit. H_S is simply the Hamiltonian of the subsystem and $\beta^{-1} = k_B T$ is the average energy of $|\Psi\rangle$.

Clearly, not all Hamiltonians yield dynamics that thermalize since free and integrable systems cannot equilibrate (if prepared in a non-thermal initial state). On the other hand, Page's theorem [3] states that a *generic* quantum state of a system with N degrees of freedom is thermal: for instance, any subsystem S with $M \leq N/2$ degrees of freedom has von Neumann entropy $S_{vN} = -\mathrm{tr}_S\rho_S\log(\rho_S) \sim \log(M)$. If $M = n^{V_S}$ where V_S is the volume of the subsystem in units of an underlying lattice and n is the number of degrees of freedom at each site, Page's theorem says that generic quantum states exhibit volume law entanglement.

[1] But less than the recurrence time.

Thus, a sufficiently *generic* Hamiltonian should thermalize any initial state. Is there a "speed limit"?

Intuitively, we expect a system can only equilibrate after all degrees of freedom have interacted with one another. For a d-dimensional system with local interactions, the time t_* for a signal to propagate across the system should scale as

$$t_* \geq cV^{z/d} \tag{0.2}$$

for some constant z. The "speed" $c = c_{z,d}$ ensures the engineering dimensions match on each side of (0.2). Generally, the speed depends on the energy contained in the system $c = c(\beta)$. For a relativistic system, $z = 1$, while diffusive propagation gives $z = 2$. In the limit $d \to \infty$ at finite z, all degrees of freedom may be understood to have interacted with one another after the time,

$$t_*(d \to \infty) \geq c(\beta) \log(V). \tag{0.3}$$

Sekino and Susskind [4] have conjectured that black holes thermalize at a time $t_{BH} \sim \beta \log(V)$ that saturates (0.3) up to an $\mathcal{O}(1)$ constant which defines the "speed limit" for thermalization to be of the order $k_B T$. In essence, any initial perturbation to the black hole is "forgotten" after the time t_{BH}.

A precise notion for the rate of thermalization[2] is provided by the exponential growth of the out-of-time-ordered (OOTO) finite-temperature correlation function [5, 6],

$$\langle \mathcal{A}(t)\mathcal{B}(0)\mathcal{C}(t)\mathcal{D}(0)\rangle_\beta \sim e^{\kappa t}, \tag{0.4}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are Hermitian operators. Thermalization occurs at a rate that is measured by the Lyapunov exponent κ. For a *generic* system, the above growth continues until

$$\frac{\langle \mathcal{A}(t)\mathcal{B}(0)\mathcal{C}(t)\mathcal{D}(0)\rangle_\beta}{\langle \mathcal{A}(t)\mathcal{C}(t)\rangle_\beta \langle \mathcal{B}(0)\mathcal{D}(0)\rangle_\beta} \sim \mathcal{O}(1). \tag{0.5}$$

Maldacena, Shenker, and Stanford (MSS) [7] have argued for the upper bound,

$$\kappa \leq \frac{2\pi k_B T}{\hbar}. \tag{0.6}$$

Thus, black holes should saturate (0.6) if the conjecture in [4] is true.

[2] We are conflating the notion of chaos and thermalization.

The acausal ordering of the operators appearing in (0.4) is the key to its utility as a theoretical probe of thermalization. The intuition behind the above OOTO correlation function arises from considering the semi-classical behavior of a particle moving in a disordered medium [6]. The OOTO correlation function for the particle's momentum,

$$-\langle [p(t), p(0)]^2 \rangle_\beta = \Big\langle \Big(\frac{\partial p(t)}{\partial x(0)} \Big)^2 \Big\rangle_\beta, \tag{0.7}$$

determines how the momentum $p(t)$ at time $t > 0$ depends on small changes of the initial position $x(0)$ at time $t = 0$. Chaos is diagnosed by an exponential divergence of the late-time behavior of states $(x(t), p(t))$ that are distinguished by small differences in the initial conditions $(x(0), p(0))$. The correlation function in (0.4) contains one term in the product of commutators $[\mathcal{A}(t), \mathcal{B}(0)][\mathcal{C}(t), \mathcal{D}(0)] = \mathcal{A}(t)\mathcal{B}(0)\mathcal{C}(t)\mathcal{D}(0) + \ldots$ and is likewise expected to reflect chaotic behavior, an important feature of thermalization [8, 9], if present.

To test the idea in (0.3) that black holes thermalize the fastest, we need a calculable model for a black hole in order to measure κ. Gauge/gravity duality [10] provides a (non-perturbative) definition for a certain class of black holes in terms of a dual quantum field theory at finite temperature. These are black holes in spacetimes described by a metric that is *asymptotically anti-de Sitter (AdS) space*. The temperature of the dual quantum field theory is equal to the gravitational force at the black hole event horizon. This remarkable correspondence relates certain questions about gravity to those of the dual field theory.

Unfortunately, there's no free lunch. Gauge/gravity duality is a "weak/strong" duality that identifies a weakly coupled gravitational theory with a strongly coupled field theory or vice versa. The best understood dual pair involves supersymmetric Yang-Mills theory on the field theory side. This theory is a supersymmetric generalization of quantum chromodynamics – loosely speaking, a cousin of the Standard Model of particle physics – that contains interactions mediated by a non-abelian gauge boson with gauge group $SU(N)$ and respects a so-called $\mathcal{N} = 4$ supersymmetry. The interactions mediated by the gauge boson are parameterized by a coupling constant proportional to N. The important point is that long-lived black holes exist in a weakly coupled gravitational theory described by classical Einstein gravity when the dual quantum field is strongly coupled $N \to \infty$. Controlled field theory calculations are generally challenging in this limit and so it is of great interest to have alternative theories that may be studied even at strong coupling.

Surprisingly, such a model arises in the theory of spin glasses [1, 11]. Consider a collection of $N \gg 1$ spins governed by the Heisenberg-type Hamiltonian,

$$H_{SY} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j. \tag{0.8}$$

The long-ranged exchange couplings J_{ij} with $i = 1, \ldots, N$ are Gaussian-random variables that couple each spin to one another. The spins $\vec{S}_i$ are chosen in a particular representation, labeled by the integer m, of the group $SU(n)$. When $n = 2$, H_{SY} governs an infinite-ranged random spin-$m/2$ system that displays spin-glass order at zero temperature [11]. This order can be suppressed in the solvable limit wherein $m, n \to \infty$ [1] (a limit in which the so-called planar or no-crossing approximation becomes exact). Sachdev and Ye found this result by decomposing the spin in terms of a product of fermions,

$$S_{\alpha\beta} = c_\alpha^\dagger c_\beta, \quad \sum_\alpha^n c_\alpha^\dagger c_\alpha = m, \tag{0.9}$$

in terms of which the Hamiltonian becomes

$$H_{SY} = \sum_{\alpha,\beta=1}^n \sum_{i,j} J_{ij} c_{i,\alpha}^\dagger c_{j,\beta}^\dagger c_{i,\alpha} c_{j,\beta}. \tag{0.10}$$

The Sachdev-Ye-Kitaev (SYK) model [1, 5] is a straightforward generalization of (0.10) to $N \gg 1$ Majorana fermions χ_i which obey $\{\chi_i, \chi_j\} = 2\delta_{ij}$ and are governed by the Hamiltonian,

$$H_{SYK} = \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l. \tag{0.11}$$

Again, the "exchange couplings" J_{ijkl} are Gaussian-random variables. All N fermions are coupled together through the 4-fermion interaction. Identifying the indices $i.j, k, l$ with points in space, the non-locality of the interaction implies the effective spatial dimension $d \to \infty$.

The 4-fermion interaction (or any polynomial interaction) grows strong at long wavelengths. For $N \to \infty$, this interaction can be reliably handled, despite the strong correlations. Disorder-averaged correlation functions display an (approximate) scale invariance[3] reminiscent of a gapless system. In particular, the Majorana fermion operators have the correlation function,

$$\langle \chi_i(t) \chi_i(0) \rangle \propto t^{-1/2}, \tag{0.12}$$

[3]This is in fact an approximate conformal invariance.

in the gapless regime [1, 5, 12]. Thus, the SYK model is an example of a strongly interacting theory that is solvable and displays an (approximate) scale invariance – two necessary requirements of a weakly curved AdS gravity dual. Kitaev [5, 13, 14] has found a Lyapunov exponent $\kappa_{SYK} = 2\pi/\beta$. Consequently, this model saturates the MSS bound (0.6). Apparently, the generalized spin system governed by (0.11) "acts" like a black hole!

We now list a few questions and comments.

• It would be interesting to calculate the Lyapunov exponent in more conventional, locally-interacting models. This exponent could provide an alternative characterization of putative many-body localized phases [15, 16].

• Are there other field theoretic systems analogous to the SYK model that can be studied reliably and offer, via duality, a probe into gravitational dynamics?

• Can κ be measured in experiments? For example, how might it be revealed in transport experiments?

• If a well-defined weakly curved AdS dual to the SYK model exists, it most naively corresponds to an embedded AdS_2 region in some geometry furnishing a UV completion [13, 14]. By duality, can we better understand how the SYK model might represent the low-energy limit of some microscopic model?

• Coupling [17] a Fermi liquid to the degrees of freedom of the Sachdev-Ye model results [18] in a marginal Fermi liquid [19]. How might such gravitational duals [20] represent other candidate non-Fermi liquids?

• The SYK model exhibits non-zero entropy density at zero temperature. A proper interpretation of this entropy density and its associated dual involving AdS_2 would be desirable.

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