No-Enclave Percolation

No-Enclave Percolation Corresponds to Holes in the Cluster Backbone
Hao Hu, Robert M. Ziff, and Youjin Deng, Phys. Rev. Lett. **117**, 185701, 2016

Recommended with a commentary by Mark Bowick and Greg Huber, Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA.

Percolation-type phenomena are widespread in physics, mathematics and computer science and serve as useful models for many physical problems. Percolation, however, comes in many types and it can be tricky to identify the precise type that is applicable in a given problem. Recently Sheinman *et al.* [1], a group based at the University of Amsterdam, studied an *in vitro* experimental cytoskeletal system composed of actin filaments, fascin cross-links and myosin motors. They observed a beautiful motor-driven collapse of the network into disjointed clusters. One of their most intriguing findings was that, over a wide range of experimental parameters, the number n_s of clusters of mass s exhibited a power-law distribution: $n_s \sim s^{-\tau}$, with $\tau \approx 1.91 \pm 0.06$. A key feature of the experiments is the apparent absence of "enclaves" — clusters fully surrounded by another cluster.

Typically, these enclaves are essential features of percolation clusters in two dimensions, and closely tied to the fractal nature of those clusters and to the fact that $\tau > 2$ in the best known class of percolation models, namely the random-bond percolation model, where $\tau = 187/91 \approx 2.05$ is the standard 2d value.

To incorporate the observed absence of enclaves into their description of actin remodeling, Sheinman *et al.* developed the so-called No-Enclave Percolation (NEP), a lattice-based model in which enclaves are absorbed into the surrounding cluster. Like the experiments, the numerics from the Amsterdam group gave a robust τ exponent less than 2. In one view, this is a cause for concern since physical quantities like the total mass of the clusters, represented as an integral over all cluster sizes, are then divergent. This raised the question of whether the size distribution of the model's clusters was skewed somehow (e.g., to favor larger clusters over smaller clusters), resulting in an underestimate of τ from a value greater than or equal to two[3].

Hu, Ziff and Deng[4] have examined this model carefully, using both theory and Monte-Carlo simulations, and have shown that it can be mapped to a problem of holes within a standard percolation backbone. This mapping successfully gives the same size-distribution exponent as seen in the NEP model $\tau = 1.82$. Hu, Ziff and Deng go further, however, and make a crucial observation: There is yet another percolation model lurking here — one with simple holes within a percolation cluster. This model predicts that $\tau = 1 + D/2 = 187/96 \approx 1.95$, where D is the fractal dimension of an ordinary 2d percolation cluster. This value agrees well with the experimental results for collapsed clusters and strongly suggests that they have identified the universality class for this system.

The theoretical arguments of Hu *et al.* are essentially sum rules that stem from two different ways of counting clusters, and they explain how τ exponents less than 2 are a natural consequence of percolation theory. These arguments were anticipated in the mid-1990s in a set of papers [5, 6] that dealt with applying analogous sum rules to both ordinary and directed percolation, as well as other examples of fractal clusters embedded within other fractal clusters. The original motivation of that work came from front de-pinning experiments, where the holes in percolation clusters and backbones represented regions between successive pinned interfaces which could be related to the dynamics of rapid, intermittent avalanches. In general, there is no divergence problem with $\tau < 2$ since there always a finite cutoff determined by the size of the surrounding cluster (or hole). The holes form a "volatile fractal" – a system which redefines itself when the size of the outside cluster grows, or when holes are subsumed into larger holes as the system size grows – a notion that goes back to work by Herrmann and Stanley on the percolation backbone[7]. One offshoot of the study in [5] was a duality relating systems with τ exponents less than 2 to those greater with τ greater than 2.

This "τ duality" relates two completely different cluster problems (call them problems A and B) and their τ exponents. The two size-distribution exponents are related as follows:

$$(\tau_A - 1)(\tau_B - 1) = 1$$

As an example, imagine that the clusters in system A are characterized by a size-distribution exponent just a tiny bit larger than 2: $\tau_A = 2 + \epsilon$. The duality formula would imply another cluster problem in system B with a τ exponent just a bit less than two: $\tau_B = 2 - \epsilon$ (to first order). Ordinary two-dimensional percolation is a good example of this phenomenon. There $\tau_A \approx 2.05$, so one would expect a dual problem with a cluster-size distribution described with $\tau_B \approx 2 - .05 = 1.95$. In fact, the value given by the above relation ($\tau_B = 187/96$) is precisely that identified by Hu, Ziff and Deng to explain the collapsed actin clusters.

Whether one calls these new exponents or a new universality class, or just new twists on percolation's known exponents, is beside the point: Hu, Ziff and Deng have pointed out that ordinary percolation continues to show its utility, including tying beautiful experimental results to sublime theoretical concepts, if only one has eyes to see it.

References

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