The Time for Disorder Has Come

Random Matrices and Complexity of Spin Glasses.
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Recommended with a Commentary by Patrick Charbonneau, Duke University

The study of spin glasses is perceived by some as hard, controversial, and useless. As a result, its fundamentals are often relegated to end-of-semester topics, prefaced by forewarnings, and surrounded by scare quotes. That is, when they are mentioned at all. Yet the last three decades have repeatedly evinced that archetypical glassy systems, such as the random energy model, the perceptron and p-spin glasses, capture the superuniversality of systems with rough energy landscapes. Beyond their original purpose as models of matter [1], these archetypes have found applications in fields in which complex landscapes are ubiquitous, including neuroscience [2], structural biology [3], and information theory [4]. And as a result, physicists familiar with disordered systems have often taken the lead in those fields.

As a counterpart to this expansion, the epistemological differences between physics and more mathematically formal fields have also been underlined. While using replicas to average over disorder and identify replica-symmetry breaking can leave some physicists uneasy, it plainly displeases electrical engineers and probabilists. The remarkable success of the replica trick as a computational scheme has thus encouraged researchers from well beyond physics to dedicate substantial efforts to confirming its predictions using alternate constructs. A number of high-profile results in this area have since come to the fore. Yet despite the elegance behind the underlying mathematics, they have not had much impact on physics.

The recent article by Auffinger *et al.* likely bucks this trend. Using the equivalence between the rough energy landscape of the spherical p-spin model and random Morse functions, this work relates the Wigner semi-circle law to the number of critical points (minima and saddle points) in the landscape. More specifically, Auffinger *et al.* used a generalization of the classical Kac-Rice formula to compute the k-complexity, $\Sigma_k(e)$, which is the logarithm of the number of critical points of degree k with energy e per spin. Although this result was already known in the physics literature (see, e.g., Ref. [5] and Fig. 1), the elegance and rigorousness of Auffinger's demonstration makes it stand out, and gives a breath of fresh air to approaches centered on the energy landscape rather than on replicas.

These ideas are potentially fruitful both for research and for pedagogy. From a research standpoint, it opens up a number of directions. First, it enables first-principle calculations of the finite-system size dynamics (in the asymptotically large size limit) within the glassy regime of mean-field models. Second, it formalizes an approach developed for studying

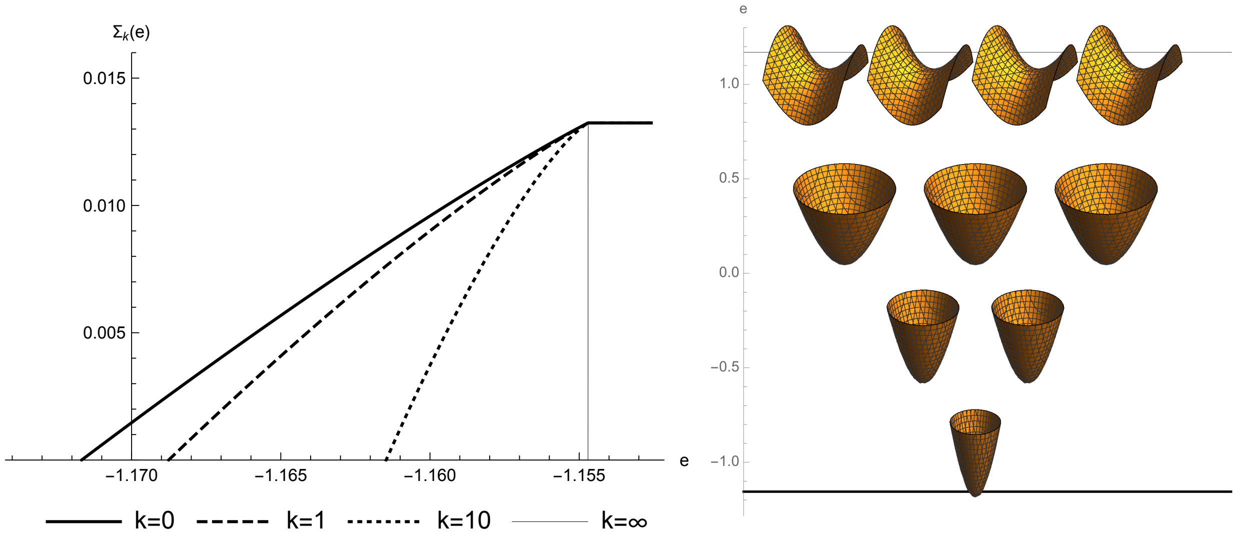

Figure 1: (left) Zero-temperature k-complexity of the spherical p-spin glass with $p = 3$. The energy minima that lie below the threshold energy, $e_{\mathrm{th}} = -\sqrt{4/3}$, are exponentially more numerous than the barriers to putatively leave them, whereas at e_{th} critical points of all orders are equally numerous. (right) Schematic of the minima in the energy landscape. Starting from the ground state energy (thick line) and going up, energy minima increase in number and widen until e_{th} (thin line), at which point essentially all minima become saddles. A system with $e < e_{\mathrm{th}}$ thus clearly remains stuck, whereas one with $e \geq e_{\mathrm{th}}$ finds many ways out. In other words, it goes from non-ergodic (glassy) to ergodic (liquid) following the mean-field random first-order transition scenario.

simpler disordered systems (Refs. [6, 7] are unfortunately not cited), which is now being expanded to complex models beyond traditional spin glasses. Third, it makes possible a replica-free estimate of the finite-temperature solution of the spherical p-spin model, taking only harmonic excitations around energy minima into account.

From a pedagogical standpoint, this work is likely even more significant. As Giulio Biroli demonstrated in a set of lectures he gave during the latest Boulder Summer School, its approach can be used to teach the elementary concepts behind the physics of disordered systems without having to toil through the more arcane aspects of the replica machinery [8]. The end result is neither hard nor controversial anymore. Because these ideas are also useful and spreading quickly, I look forward to encountering (under)graduate statistical mechanics textbooks that embrace them.

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