# From local observables in a single eigenstate to parent Hamiltonians

**Determining a local Hamiltonian from a single eigenstate**
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arXiv:1712.01850

*Recommended with a Commentary by Ashvin Vishwanath, Harvard University*

In contrast to the standard approach, of beginning with a Hamiltonian and proceeding to find its eigenstates, the featured reference invites us to consider the inverse problem. Given a *single* eigenstate, can we reverse engineer the corresponding Hamiltonian of a many body system? The answer to this question, modulo certain caveats, is argued to be yes. Given the enormous amount of information packaged in a many-body wavefunction this may not be surprising, and might seem to be of limited utility. However the reference goes beyond this and argues that even just knowing a set of two point correlation functions, evaluated in this single eigenstate, is sufficient to reverse engineer the Hamiltonian.

A crucial ingredient here is that the assumption that the Hamiltonian is *local* i.e. that interactions decay rapidly with separation. Otherwise, one could simply use the projector on the eigenstate as a Hamiltonian, while leaving its action on orthogonal states arbitrary. These operators would be extremely nonlocal, and are eliminated by the restriction to local Hamiltonians.

More concretely, the physical system one has in mind is a lattice system of spins, where the Hilbert space on each site is finite (say 'm' dimensional, which for spin 1/2 per is m=2). Further, locality of the Hamiltonian will be implemented by requiring that interactions have a finite range $k$ in lattice units, and all interactions with longer range are simply zero. So, with just on-site and nearest neighbor coupling we have $k = 2$. For simplicity we imagine a 1D system, although the results hold in arbitrary dimension. Say we are given a single eigenstate $|\psi\rangle$ (either the ground state or an excited state) of a 'k' local Hamiltonian on this lattice. The key idea is that one measures the correlators of the set of 'k' local operators $O_a(r)$ centered at $r$ and labelled by $a$ (collectively labeled $\alpha = (a, r)$). On then calculates the correlation Matrix $[\mathbf{M}]_{\alpha\beta} = \langle\psi|O_\alpha O_\beta|\psi\rangle - \langle\psi|O_\alpha|\psi\rangle\langle\psi|O_\beta|\psi\rangle$. This matrix is clearly Hermitian, but can also be shown to be positive semidefinite (i.e. all eigenvalues are non-negative). If we have included all local operators that could potentially appear in the Hamiltonian, that can also be expanded as: $H = \sum_\alpha c_\alpha O_\alpha$. The aim then is to find the coefficients in this expansion. It can be readily shown that the Hamiltonian corresponds to the zero eigenvector of the correlation matrix, i.e. in matrix notation the coefficients satisfy $\mathbf{Mc} = 0$. This follows from

observing that the quantity $\mathbf{c}^\dagger \mathbf{M} \mathbf{c}$ is related to the fluctuations of the total energy in the state $|\psi\rangle$, which vanishes for an eigenstate.

Notice, the set of parameters characterizing a local Hamiltonian is 'small' compared to the number of parameters in a typical wavefunction. In a spin 1/2 chain with just on site and nearest neighbor couplings, there are three nontrivial onsite operators $\sigma^a$, the three Pauli matrices and nine two site operators $\sigma_i^a \sigma_{i+1}^b$, where $a$, $b = 1$, 2, 3. Hence, in all there are 12 matrices. So, in a chain with $N$ sites, we need $12N$ parameters to specify the most general such Hamiltonian (and significantly less if we assume translation invariance). In contrast, the number of complex numbers entering a wavefunction is exponentially large in the size: $2^N$. Hence, most of the information in the wavefunction is redundant for this purpose and one can distill the requisite information from the correlation matrix.

An important question pertains to the uniqueness of the solution. In several cases one would actually expect more than one null eigenvector, leading to a family of solutions. For example, in a system with a conserved charge, the operator measuring the total charge $N$ will, like the Hamiltonian, have no fluctuations in an eigenstate and is also local. Adding $\mu N$ to the Hamiltonian does not change its eigenstates and hence must be admitted. The possibility of accidental degeneracies is more serious and can appear in certain cases such as in many body localized with infinitely many local conserved quantities.

This technique may be useful for obtaining parent Hamiltonians from variational ground states that model exotic states of matter, and may ultimately help in the search for such phases. Also, for translationally invariant Hamiltonians, relatively few measurements are required. The measurements will however involve higher order correlators such as $\langle \sigma_i^z \sigma_{i+1}^z \sigma_j^z \sigma_{j+1}^z \rangle$ and an interesting question is if such measurements are feasible say using a quantum gas microscope. Another important question is if deviating from the ground state, for example because of finite temperature, would lead to significant errors when reconstructing the Hamiltonian.

Other related works have argued [1] how one can use the eigenstate thermalization hypothesis to extract the Hamiltonian from of a single excited state, although information beyond two point correlators are needed. In Ref. [2], an algorithm to construct ground states from the values of local observables was discussed.

The key problem of many body physics, of starting from a local Hamiltonian and predicting expectation values of local observables (for example in the ground state) is a map from a 'small' vector $c_\alpha$, (linear in system size) to a small matrix (like $M$). Nevertheless one has to take a diversion into an exponentially large space, the space of eigenstates, to achieve this. In some situations, Matrix Product states and other tensor methods can tame this exponential. The featured reference shows that the inverse problem, of recovering $H$ from $M$ does not require this diversion.

In Quantum Many Body physics it appears that if you know the answer - some properties evaluated in an eigenstate - it is typically easy to go back to the question, i.e. the Hamiltonian. This is apparently not the case for some other problems - for example in Ref. [3] the answer, 42, was significantly easier to compute than the question.

I thank Yi Zhuang You and Adrian Po for helpful discussions.

# References

[1] Does a single eigenstate encode the full Hamiltonian? James R. Garrison, Tarun Grover. arXiv:1503.00729.

[2] Reconstructing quantum states from local data. Brian Swingle, Isaac H. Kim. Phys. Rev. Lett. 113, 260501 (2014).

[3] Douglas Adams, The Hitchhiker's Guide to the Galaxy, Simon & Schuster (1979).