# Successful Engineering of a Yang Monopole

**Observation of a non-Abelian Yang Monopole: From New Chern Numbers to a Topological Transition**
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*Recommended with a Commentary by Tin-Lun Ho, The Ohio State University*

In 1931, P.A.M Dirac pointed out that the total magnetic flux emerging from a magnetic monopole must be quantized as a consequence of the single-valued-ness of the electron wave function[1]. The Dirac monopole is an example of an Abelian monopole, as it is a solution of a gauge theory (quantum electrodynamics) with Abelian gauge group U(1). Half a century later, C.N. Yang generalized this solution to the non-Abelian SU(2) gauge theory in *five* dimensional Euclidean space [2]. A gauge field $\mathbf{A} = (A_1, A_2, , , , A_n)$ in $n$-dimensional Euclidean space is non-Abelian if the different components fail to commute. The resulting curvature

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial r_\nu} - \frac{\partial A_\nu}{\partial r_\mu} + i[A_\mu, A_\nu] \quad (1)$$

is the analog of the electromagnetic field strength in electrodynamics. In three spatial dimension, the magnetic field is given by $B_\lambda = \epsilon_{\lambda\mu\nu}F_{\mu\nu}/2$. The magnetic flux through a spherical surface $S_2$ is given by the first Chern number

$$C_1 = \frac{1}{2\pi}\int_{S_2} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \frac{1}{4\pi}\int_{S_2} F_{\mu\nu}\mathrm{d}r_\mu \wedge \mathrm{d}r_\nu. \quad (2)$$

The $n$-th Chern number is obtained by replacing the integrand in Eq. (2) with the $n$-th Chern form (proportional the $n$-th power of the curvature matrix $F$) integrated over a 2n-dimensional sphere $S_{2n}$. While the Dirac monopole has $C_1 \neq 0$, Yang monopole has $C_1 = 0$ and $C_2 \neq 0$.

Dirac monopole is responsible for many phenomena in condensed matter. Its presence in a band structure leads to a quantized Hall conductance given by the monopole charge $C_1$. Efforts to find the non-Abelian monopoles have led to studies of four- (and higher) dimensional crystals. It has been shown that a 4D crystal with $C_2 \neq 0$ will have a quantized non-linear electro-magnetic response[3]. The study of higher dimensional crystals is not purely academic, as the Floquet spectrum of a 3D crystal in an oscillating field can be viewed as the band-structure of a 4D crystal. Moreover, recent experiments in cold atoms can create higher dimensional solids using the internal degrees of freedom of atoms as "synthetic" dimensions. The engineering a Yang monopole therefore opens the door to study topological phenomena in higher dimensions, and their influence on the physics in lower dimensions.

In a recent paper, Ian Spielman's group at NIST has reported the creation of a Yang monopole using a Bose-Einstein condensate of Rb atoms. The idea can be illustrated by a "spin" analog of the monopole. It was pointed out by Michael Berry[5] that the quantization of the flux of a magnetic monopole[1] can also be realized in the simple example of a spin-1/2 particle in a magnetic field $\mathbf{b}$, with Zeeman energy $H = -\mathbf{b} \cdot \mathbf{S}$. Berry showed that if the "connection" $\mathbf{A} = -i\langle G|\vec{\nabla}|G\rangle$, $(\nabla_i = \partial/\partial b_i)$, is regarded as an effective "gauge field", the associating first Chern-number is precisely the quantized magnetic flux of Dirac. For systems with degenerate ground states, the connection $\mathbf{A}_{\alpha\beta} = -i\langle\alpha|\vec{\nabla}|\beta\rangle$ is a matrix and is in general non-Abelian. The simplest spin system with this feature is a spin 3/2 atom in an electric field, with Stark hamiltonian $H_{ED} = \sum_{i,j=x,y,z} Q_{ij} J_i J_j$, where $Q_{ij}$ is a real traceless symmetric matrix, which has *five* independent parameters. Since $H$ is invariant under time reversal, Kramer's theorem implies that the ground state must be two-fold degenerate, $(\alpha = 1, 2)$. As it turns out, $H_{ED}$ can be rewritten in the conventional relativistic form of a Yang monopole, $H_Y = -\sum_{\mu=1}^{5} k_\mu \Gamma_\mu$, where $\{k_\mu\}$ are five real parameters related to those in $Q$, and $\Gamma_\mu$ are the $4 \times 4$ Gamma matrices. In other words, the ground states of $H_Y$ form a doubly degenerate manifold in a 5D parameter space. The fact that $C_1 = 0$ for the Yang monopole is also a consequence of Kramer degeneracy.

Spielman's group at NIST set out to simulate the $4 \times 4$ matrix $H_Y$ using the four spin states of Rb-87 atoms. They generate the off diagonal matrix elements by coupling different spin states with Raman processes, and the diagonal matrix elements using magnetic fields and light fields[4]. To show that the engineered Hamiltonian has the expected Chern-number, they first determine the curvature matrix $F_{\mu\nu}$ experimentally. This is done by measuring the response of the matrix $\delta\langle\Gamma_\mu\rangle$ with respect to the change of $\delta k_\nu$, as it can be shown $\delta\langle\Gamma_\mu\rangle = F_{\mu\nu}\delta k_\nu$ [6]. Once the curvature tensor is determined, $C_2$ can be evaluated (in principle) by numerically integrating $F$ over a 4D spherical surface. This is a daunting task. To make it feasible, the authors have engineered a parameter set $k_\mu$ so that $H_Y$ has very high symmetry in the parameter space. In this way, the 4D integral is reduced to one in 2D, whose evaluation is experimentally manageable. With this procedure, the NIST group has found that $C_1 = 0$ and that $C_2$ changes form 1 to 0 as the monopole moves outside the surface of integration, showing a successful simulation of the Yang monopole.

The success of this simulation paves the way for engineering quantum manifolds of even greater complexity. It also provides a convenient setting to study interaction effects on monopoles, which are particularly dramatic for bosons. Since the "Zeeman" energy vanishes at the monopole singularity, the quantum state near the singularity is always dominated by interactions, which can have a different Chern number. It is certainly interesting to study similar effects for fermions. The ability to create complex quantum manifold and change interactions in cold atom experiments will help to shed light on many new problems involving both interaction and topology.

## References


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