Drift or diffusion? Time periodic drive versus time correlated drive

Langevin Dynamics with Space-Time Periodic Nonequilibrium Forcing
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The purpose of this comment is to attract attention to two groups of possibly related results, each of which is rather well known, but apparently in non-overlapping communities.

R. Joubaud with co-authors, in their 2015 J. Stat. Phys. paper, following the 2008 work of P.Collet and S.Martínez [1], examine, in a mathematically rather sophisticated way, the following simple (simply formulated!) problem. Consider a particle diffusing in 1D subject to a potential which is a periodic function of both space and time:

$$\partial_t P = D\partial_x^2 P + \partial_x \left[b(t,x)P \right] \ , \tag{1}$$

where drift velocity $b(x,t)$ is such that $b(x+L,t) = b(x,t)$ and $b(x,t+T) = b(x,t)$; in appropriate units, $b(x,t) = -(1/\zeta)\partial_x U(x,t)$, with U potential, and $1/\zeta$ mobility. If potential is time-independent and spatially periodic, the particle undergoes an unbiased diffusion, even if potential is asymmetric (like, e.g., a saw tooth). The central statement is that time-periodic change of potential may result in developing a drift in a certain direction asymptotically in the limit of long observation time (longer than period T). This is true even when function $b(x,t)$ is zero on spatial average, $\int_0^L b(x,t)dx = 0$. It is obvious on physical grounds that such an asymptotic drift may only exist if potential is spatially asymmetric (like a saw tooth, not invariant under $x \leftrightarrow \mathrm{const} - x$), although mathematicians never mention this trivial fact in their papers.

Furthermore, R. Joubaud with co-authors add also a "ramp" potential fx, replacing $U(x,t) \to U(x,t) + fx$, and show that the particle may even drift up the "ramp", if f is small enough (and has appropriate sign).

This phenomenon is studied by mathematicians in a rather formal way. It should be compared with the result that goes back to the paper by M.Magnasco of 1993 [2] and that was refined in a number of later works (see [3] for example and for further references). This one has to do with a particle moving in a spatially periodic and time-independent potential $U(x)$ driven by time-correlated colored noise, for instance, Ornstein-Uhlenbeck noise:

$$\zeta\partial_t x = -\partial_x U(x) + \eta(t) \ , \quad \langle \eta(t) \rangle = 0 \ , \quad \langle \eta(t)\eta(t') \rangle = \frac{k_B T \zeta}{\tau} e^{-|t-t'|/\tau} \ . \tag{2}$$

In the white noise limit ($\tau \to 0$), this Langevin equation describes pure unbiased diffusion, whether $U(x)$ is symmetric or not. But as soon as $\tau > 0$, there appears a drift provided the potential is not symmetric. In this sense, time-correlated noise appears to have the same effect as time periodic drive.

It seems a worthy question to understand if there is a deep underlying mathematical commonality for these effects. Note that the first problem is easier to formulate in the language of Fokker-Planck equation (1), while Langevin equation for that problem is non-linear and does not look easy. By contrast, second problem is most naturally stated in terms of Langevin equation (**??**) while writing a Fokker-Planck equation for this problem requires a trick. Also, it might be a worthy exercise to play the game of R. Joubaud and co-authors for the system (2): to add a ramp potential in equation (2) and see if time-correlated driving noise is strong enough to force the system to climb up the ramp.

References

[1] P.Collet, and S.Martínez, J. Math. Biol. **56**(6), 765792 (2008)

[2] M. O. Magnasco, Phys. Rev. Lett. **71**, p. 1477 (1993).

[3] C.Sandford, A.Y. Grosberg, J.-F. Joanny Phys. Rev. E **96**, 052605 (2017)