Nonintegrable mechanics

Odd elasticity
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Recommended with a Commentary by Aparna Baskaran, Brandeis University

A quote often attributed to John von Neumann is that the theory of nonequilibrium systems is like the theory of non-elephants. Hence, the way we make progress in theoretical understanding of nonequilibrium phenomena is to take one attribute of the equilibrium elephant, change it in a particular way and map out the consequences of this change to the dynamics and statistical steady states of the corresponding non-equilibrium system (a particular non-elaphant). In this commentary, let us focus on one aspect of the theory of an equilibrating macroscopic system, namely that there exists a free energy functional that drives the dynamics of the system.

For example, suppose the relevant macroscopic variable for the dynamics of a particular system is a conserved quantity ϕ. In a equilibrating system the dynamics of this variable is given by $\partial_t \phi = \nabla^2 \frac{\delta F}{\delta \phi}$, where simplest form of the free energy functional is of the form

$$F = \int d\mathbf{r} \left[a_2 \phi^2 + a_4 \phi^4 + \kappa(\rho)(\nabla \phi)^2 \right]$$

This particularly simple integral form of the dynamics emerges due to detailed balance and time reversal symmetry at the level of the microdynamics of the system and is variously referred to as Model B dynamics, phase field dynamics or the Cahn-Hilliard equation. It robustly describes phase separation phenomena in passive colloidal systems including the celebrated universal scaling of domain sizes during coarsening, $L \sim t^{1/3}$.

In an inherently nonequilibrium system, there is no a priori reason to expect the dynamics to be integral. A simple non-integrable version of this scalar field theory first considered in [1] is of the form

$$\partial_t \phi = \nabla^2 \frac{\delta F}{\delta \phi} + \frac{1}{2} \nabla^2 (\kappa'(\phi)(\nabla \phi)^2)$$

and helped us understand athermal phase separation in self-propelled particle systems, where internal propulsion of active agents with purely repulsive interactions drives a gas-liquid transition.

Similarly, if a system was described by a non-conserved vector order parameter $\mathbf{P}$, (for example an equilibrating ferrofluid) it would exhibit integral dynamics of the form $\partial_t \mathbf{P} = -\frac{\delta F}{\delta \mathbf{P}}$, with a free energy functional of the form

$$F = \int d\mathbf{r} \left[a_2 |\mathbf{P}|^2 + a_4 |\mathbf{P}|^4 + \kappa(|\mathbf{P}|)(\nabla \cdot \mathbf{P}) + \frac{K}{2} (\nabla \mathbf{P})^2 \right]$$

If we break the integrability and write down a particular nonintegrable dynamical equation of the form

$$\partial_t \mathbf{P} = -\lambda \mathbf{P} \cdot \nabla \mathbf{P} - \frac{\delta F}{\delta \mathbf{P}}$$

we obtain the celebrated Toner-Tu equations that demonstrated the existence of long range order with continuous symmetry breaking in 2D flocking systems [2, 3].

The featured paper in this commentary considers the consequence of non-integrable dynamics in the context of a driven linear elastic material. To formulate the problem in the language above, let us consider a linear elastic solid whose dynamics is overdamped. In this case, we can write its equation of motion as

$$\partial_t u_{ij} = \frac{\delta F}{\delta u_{ij}}$$

where u_{ij} is the symmetrized strain tensor and $F = \int d\mathbf{r} \frac{1}{2} C_{ijkl} u_{ij} u_{kl}$ is the elastic free energy of the system. Due to the integral nature of the dynamics, it is easy to see that the stress-strain relationship $\sigma_{ij} = K_{ijkl} u_{kl}$ is characterized by a stiffness tensor K that has the symmetry property $K_{ijmn} = K_{mnij}$ in addition to any other constraints placed by the symmetries of the material itself. What if we had a driven material for which we postulate a linear dynamics of the form $\partial_t u_{ij} = K_{ijkl} u_{kl}$ that in principle could be nonintegrable, i.e., K does not have the symmetry property with respect to interchange of indices implied by energy conservation? This is the question addressed by the authors of the featured paper.

The paper takes a pedagogical approach to demonstrate that such a non-integrable mechanics leads to work being done by an elastic cycle. Focusing on the particular case of an isotropic solid with no energy conservation, the authors show the emergence of auxetic behavior and propagating modes in this overdamped system that are powered by the self-sustained elastic cycles set up by the driving. These considerations identify one other candidate in this class of non-elephants and offer design principles for building a class of driven materials that are powered by elastic work cycles.

References

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