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How to measure the Euler characteristic of the Fermi sea

Quantized Nonlinear Conductance in Ballistic Metals
Authors: C. L. Kane
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Recommended with a Commentary by Carlo Beenakker, Instituut-Lorentz, Leiden University

This recent paper by Charlie Kane struck me because it starts from an effect I thought was completely understood and restricted to 1D, then finds a novel interpretation which suggests a higher-dimensional generalization. The 1D effect is the quantized conductance, $G = Ne^2/h$, of an N-mode wire without any disorder (ballistic transport). The novel interpretation is that the integer N is the Euler characteristic χ of the 1D Fermi sea, and the generalization is that in d dimensions χ governs the nonlinear conductance of a $d + 1$ terminal geometry.

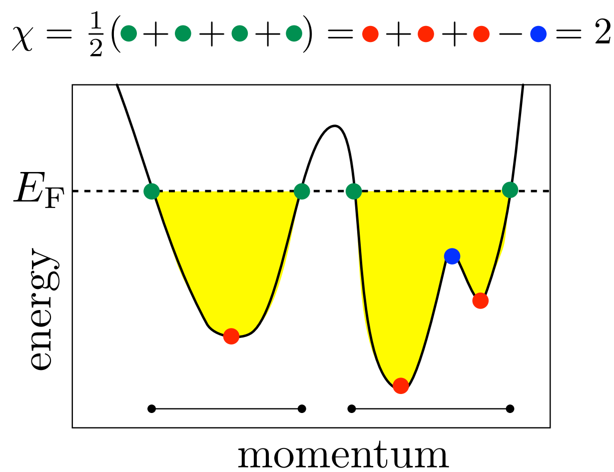

Fig. 1. The 1D Fermi sea (yellow regions) of this dispersion has Euler characteristic $\chi = 2$ (two line segments along the momentum axis). The same number can be obtained by counting the number of intersections of the dispersion with the Fermi level (green dots) or by counting the number of extrema below the Fermi level (red and blue dots). The ballistic conductance equals $G = Ne^2/h$ with $N = \chi$.

The $d = 1$ case is illustrated in Figure 1. To find the conductance of a two-terminal ballistic wire one would count the number of intersections of the dispersion relation $E(k)$ with the Fermi level $E_{\rm F}$. Each of the N intersections with positive slope identifies a right-moving propagating mode, which contributes e^2/h to the conductance. The same number N counts the number of intervals of momentum k with $E(k) < E_{\rm F}$. These k-intervals form the Fermi sea, the filled states below the Fermi level. Since a line segment has Euler characteristic $\chi = 1$ (2 vertices minus 1 edge plus 0 faces = 1), the number N can also be called the Euler characteristic χ of the 1D Fermi sea.[1]

[1] In this 1D case one might equivalently identify N with one half the Euler characteristic $\tilde{\chi}$ of the Fermi *surface* $\{\boldsymbol{k}|E(\boldsymbol{k}) = E_{\rm F}\}$. More generally, for odd d one has $\tilde{\chi} = 2\chi$, while $\tilde{\chi} = 0$ for even d, hence the focus on the Fermi sea rather on its boundary, the Fermi surface.

True, but why would you want to do that? As Kane explains, with this identification one can think about the generalization to a higher dimensional Fermi sea. For that purpose it is helpful to use yet another way to extract χ from the dispersion relation $E(\boldsymbol{k})$: First identify the set of critical points, the momenta $\boldsymbol{k}$ in the first Brillouin zone where $E < E_{\rm F}$ and $\nabla E = 0$. Group these points into sets of size $N_{\rm even}$ and $N_{\rm odd}$, depending on whether $E(\boldsymbol{k})$ decreases along an even or an odd number of axes as one moves away from the critical point. Then Morse's theorem says that $\chi = N_{\rm even} - N_{\rm odd}$. In 1D there is only one axis, so $N_{\rm even}$ counts the number of minima and $N_{\rm odd}$ the number of maxima of the dispersion. In 2D, with two axes, both minima and maxima contribute to $N_{\rm even}$, while $N_{\rm odd}$ counts the number of saddle points below the Fermi level.

As an example, Figure 2 shows the dispersion relation $E(k_x, k_y)$ for a 2D square lattice. When the Fermi level is just above the band bottom, the Fermi sea is a disc, there is one minimum below $E_{\rm F}$ so $N_{\rm even} = 1$, $N_{\rm odd} = 0 \Rightarrow \chi = 1$. Check with Euler's polyhedron formula: the disc can be deformed into a triangle, and 3 vertices minus 3 edges plus 1 face = 1.

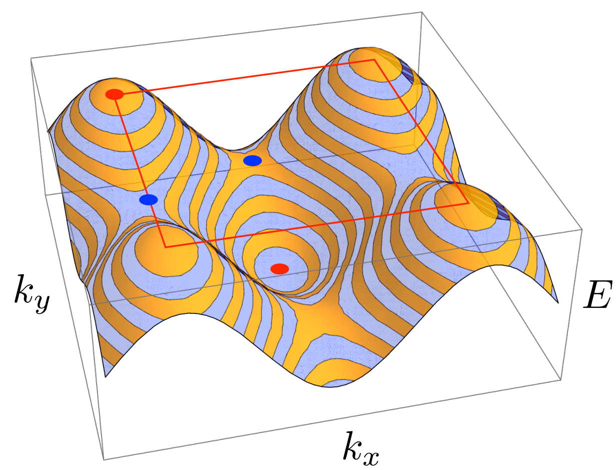

Fig. 2. Equi-energy contours of the 2D dispersion $E(k_y, k_y) = 3 - 2\cos k_x - \cos k_y$. The red square indicates the Brillouin zone. With increasing $E_{\rm F}$ the topology of the Fermi sea changes from a disc ($\chi = 1$), to a cylinder ($\chi = 0$), to a torus minus a disc ($\chi = -1$), and finally to a torus ($\chi = 0$) for a fully filled band. The inequivalent critical points are indicated by red dots (one minimum and one maximum, contributing to $N_{\rm even}$) and blue dots (two saddle points, contributing to $N_{\rm odd}$).

If $E_{\rm F}$ is increased it crosses a saddle point, so $N_{\rm odd}$ increases by one unit and $\chi = N_{\rm even} - N_{\rm odd} = 0$. The Fermi sea then extends over the Brillouin zone in one direction, and since opposite edges of the Brillouin zone must be identified the topology is that of a cylinder. Upon further increase of $E_{\rm F}$ a second saddle point is crossed and when the Fermi level crosses the top of the band one has $N_{\rm even} = N_{\rm odd} = 2 \Rightarrow \chi = 0$. The Fermi sea then extends over the Brillouin zone in two directions, it has the topology of a torus.

So how would one measure this topological quantum number? We are considering a gapless system, with a partially filled band, so the case for topological protection is less favorable than it is, for example, in the case of the Chern number of a filled band (measured via the quantum Hall effect). The Chern number is an integral over the Berry curvature, and the Euler characteristic can likewise be expressed as an integral over the geometric curvature, but this analogy has not yet inspired a measurement scheme.

Kane takes a different approach, relying on the relationship $\chi = N_{\rm even} - N_{\rm odd}$ between the Euler characteristic and the critical points of the dispersion relation. Figure 3 shows a $d+1$ terminal geometry in 2D and in 3D. The d-dimensional solid angle is divided into $d+1$ sectors, voltages $V_1, V_2, \ldots V_d$ are applied to the first d sectors while the last sector draws a current I to ground. A sequence of voltage pulses of integrated area $\int V_d(t)\,dt = h/e$

transfers one state through each critical point in the dispersion, producing a current pulse $\int I(t)dt = \chi e$. In the frequency domain, for $V_d(t) = V_d \cos(\omega_d t)$, the current to ground contains a component at frequency $\omega_\Sigma = \sum_{n=1}^{d} \omega_d$ given by

$$I(\omega_\Sigma) = \chi e\, \omega_\Sigma \prod_{n=1}^{d} \frac{eV_d}{h\omega_d}. \tag{1}$$

I must admit that I have not yet succeeded in convincing myself of the generality of this formula. I have no reason to doubt the calculations in Kane's paper, a semiclassical Boltzmann equation is sufficient to derive Eq. (1), or one can apply the quantum nonlinear response theory. But the calculations are based on a specific geometry, and it is not obvious to me how general the answer is. One example I am struggling with: take the 3D setup in Figure 3 and rotate the lower half of the sphere by 90° around the z-axis. The planes that separate the sectors then meet in a point rather than along a line. Does it matter?

This is actually the reason I'm recommending this paper in the journal club: It opens up a line of thought that promises fundamental new insights into transport properties of complex band structures. Are there other observables that provide a global measurement of the Euler characteristic of the Fermi sea?

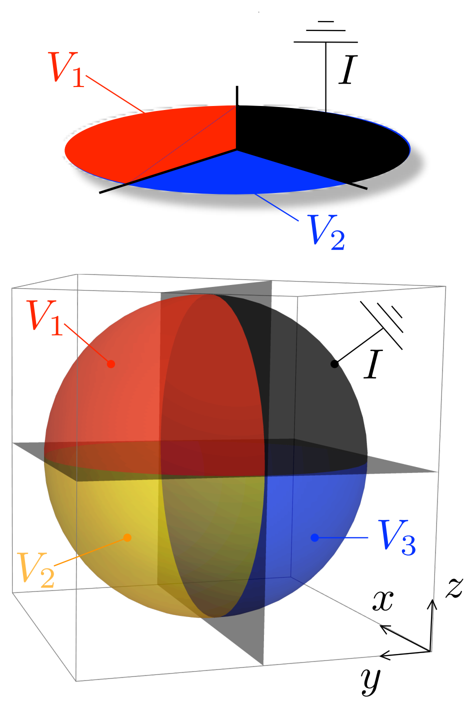

Fig. 3. Two-dimensional conducting disc (top, $d = 2$) and three-dimensional conducting sphere (bottom, $d = 3$), with d voltage contacts V_n and one contact to ground.