A better large N limit of the electron-boson problem?

Large N Theory of Critical Fermi Surfaces
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The electron-boson problem – i.e. the problem of a dense electron fluid interacting via a Yukawa coupling with a bosonic field which could represent phonons or the collective fluctuations of an order parameter near a quantum critical point – is one of the key problems in effective field theory to our understanding of condensed matter systems. When the coupling is weak, this problem can be treated straightforwardly using the machinery of diagrammatic perturbation theory. But when the dimensionless coupling, λ, is of order one (as it presumably is in most physically relevant circumstances), this is an example of a strongly correlated problem, and is highly non-trivial. The Migdal approximation has been claimed to hold true so long as $\lambda \ll \sqrt{M/m}$ (where M is an ion mass and m is the electron mass), but it is by now well documented that this is not generally true.[1] Now, Esterlis *et al*, in the highlighted paper, have shown that the Migdal approximation is exact in the $N \to \infty$ limit of an appropriate generalization of the electron-boson problem to the case in which there are N symmetry-related flavors of fermions and the same number of bosons. This may open the door to well controlled studies of a number of interesting problems – including the properties of near quantum critical metals – using already well-developed theoretical technologies.

Needless to say, there have been previous studies that have explored various large N limits of this problem:

The most common involves N flavors of fermions – an approach that reproduces the familiar Hertz-Millis theory when applied to the problem of metallic quantum critical points. At a technical level, it has been shown in Ref. [2] that the 1/N corrections to this theory are singular, particularly in the important case of $d = 2$ spatial dimensions. At a physical level, this limit has the peculiar property that the bosons are highly dressed by their coupling to the fermions ("Landau damping"), but the fermions remain the long-lived quasi-particles of a Fermi liquid. For finite N, it can be easily shown that the scattering of the phonons from Landau-damped quantum critical bosons destroys the Fermi liquid. Thus, at a physical level, there is a lack of self-consistency to this approach. Clever methods extending the large N approach with non-local interactions have been proposed to remedy this issue,[3] but the presence of non-local interactions is an additional serious modification to the microscopic problem, making the relevance to any particular situation less transparent.

A more recent "matrix large N" approach to the problem was introduced in Ref. [4]; here there are N flavors of fermion and N^2 bosons. This limit has the opposite problem – at $N \to \infty$ it gives rise to a non-Fermi liquid but includes no feedback of the coupling on the bosonic sector. Since Landau damping clearly arises for any finite N, here, too, there is a physical sense in which the large N limit is singular. Again, clever approaches to addressing this issue have been introduced[5], involving taking the limit that the radius of the Fermi surface, $k_F \to \infty$ as $N \to \infty$.

A significant feature of the approach presented in the Esterlis et al paper is that the large N limit is simple and physically reasonable. It reflects corresponding effects on both the fermions and the bosons, through the familiar self-consistent Migdal integral equations for the boson and fermion self-energies. No pathologies – such as a violation of the third law of thermodynamics – arise as $N \to \infty$.

A first key step in defining the model was introduced independently in two recent papers, by Esterlis and Schmalian[6] and by Wang[7] (ESW). Here, in a zero-dimensional (single site) version of the model, the Yukawa coupling is taken to be a random three-index flavor-tensor, $g_{a,b,c}$. In the large N limit, it is argued that the properties of the system are self-averaging, so one can perform an average over these quantities. This construction was generalized to higher dimension in Refs. [8], and [9], although in a slightly different context than that treated by Esterlis *et al.* It is important to note that all these higher dimensional versions are not models with disorder; while as in the ESW single site problem, $g_{a,b,c}$ is assumed to be random, it is taken to be translationally invariant, i.e. the coupling is the same on every site. [10]

The nature of the solution of the resulting integral equations depends on the assumed properties of the boson involved. For bosons with the character of optical phonons, these equations yield standard results, such as the Bloch-Grunheisen expression for the resistivity, and – extended to allow for an anomalous propagator – the Migdal-Eliashberg theory of the superconducting state. On the other hand, the equations are highly non-linear and there is far from an exhaustive understanding of the full set of possible self-consistent behaviors. This applies in particular to non-Fermi liquid solutions that might represent various classes of quantum critical phenomena.

As with any such approach, it remains to be determined to what extent the properties of the $N \to \infty$ problem capture the essential features of the problem with physically relevant values of N – especially $N = 1$. While there are no clear physical pathologies associated with the $N \to \infty$ limit of the model, there has not yet been any successful analysis of the leading $1/N$ corrections. In the context of quantum critical phenomena, it is also unclear how to think about the $O(N)$ symmetry – which is explicitly broken by typical values of $g_{a,b,c}$, and is only realized on average. It seems important to explore more fully the relation between a system with a true symmetry and with only a statistical symmetry of an ensemble. On the other hand, the ESW model has the great advantage that the $N \to \infty$ analysis is straightforward, yields explicit expressions for a large range of physical properties, and is easily generalized to include all sorts of other effects.

References

[1] For a perspective on the breakdown of the Migdal approximation, see A. V. Chubukov, A. Abanov, I. Esterlis, and S. A. Kivelson, “Eliashberg theory of phonon–mediated superconductivity – When it is valid and how it breaks down,” Ann. of Phys. **417**, 168190 (2020).

[2] S-S. Lee, “Low-energy effective theory of Fermi surface coupled with U(1) gauge field in 2+ 1 dimensions,” Phys. Rev. B **80**, 165102 (2009).

[3] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, “Controlled Expansion for Certain Non-Fermi-Liquid Metals,” Phys. Rev. B **82**, 045121 (2010)

[4] A. L. Fitzpatrick, S. Kachru, J. Kaplan, and S. Raghu, “Non-Fermi-liquid behavior of large N_B quantum critical metals,” Phys. Rev. B **89**, 165114 (2014).

[5] J. A. Damia, S. Kachru, S. Raghu, and G. Torroba, “Two-Dimensional Non-Fermi-Liquid Metals: A Solvable Large-N Limit,” Phys. Rev. Lett. **123**, 096402 (2019).

[6] I. Esterlis and J. Schmalian, “Solvable Strong-Coupling Quantum-Dot Model with a Non-Fermi-Liquid Pairing Transition,” Phys. Rev. B **100**, 115132 (2019).

[7] Y. Wang, “Solvable strong-coupling quantum-dot model with a non-Fermi-liquid pairing transition,” Phys. Rev. Lett. **124**, 017002 (2020).

[8] E.E. Aldape, T. Cookmeyer, A. A. Patel, and E. Altman, “Solvable theory of a strange metal at the breakdown of a heavy Fermi liquid,” arXiv:2012.00763 (2022).

[9] J. Kim, E. Altman, and X. Cao, “Dirac Fast Scramblers,” arXiv:2010.10545 (2021).

[10] A similar construction – involving a periodic array of identical SYK dots – was introduced earlier in D. Chowdry, Y. Werman, E. Berg, and T. Senthil, “Translationally Invariant Non-Fermi-Liquid Metals with Critical Fermi Surfaces: Solvable Models,” Phys. Rev. X **8**, 031024 (2018).