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Spin Liquids by Measurement Alone

1. **Monitored Quantum Dynamics of the Kitaev Spin Liquid**
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   arXiv:2207.02877

2. **Topology, criticality, and dynamically generated qubits in a stochastic measurement-only Kitaev model**
   Authors: Adithya Sriram, Tibor Rakovsky, Vedika Khemani, and Matteo Ippoliti
   arXiv:2207.07096

Recommended with a Commentary by Daniel Arovas, University of California, San Diego

Kitaev's celebrated model of a $S = \frac{1}{2}$ spin-liquid with two-body interactions on the honeycomb lattice [1] is a beautiful example of fractionalization. The Hamiltonian is deceptively simple: $S = \frac{1}{2}$ spins on each site of the honeycomb lattice, interacting with their nearest neighbors via XX', YY', or ZZ' Ising interactions, according to the bond type $\alpha \in \{x, y, z\}$. Then for every hexagonal plaquette p, the product W_p of spin operators, as shown in fig. 1, commutes with all terms in H, as well as with all other $W_{p'}$, which may then be fixed[1]. In Kitaev's solution, the spin operators are expressed in terms of four Majorana fermions, with $\sigma^\alpha = icb^\alpha$. The constraint $cb^x b^y b^z = -iXYZ = 1$ at each site guarantees that $XY = iZ$ and that the local Hilbert space is two-dimensional. In terms of the Majoranas,

$$H = \sum_\alpha J_\alpha \sum_{\langle ij \rangle}^{\text{type}-\alpha} iu_{ij}\, c_i\, c_j \tag{1}$$

where each $u_{ij} = -ib_i^{\alpha_{ij}} b_j^{\alpha_{ij}} = \pm 1$ is a $\mathbb{Z}_2$ gauge field and thus the interacting spin Hamiltonian may be written in terms of a single species of noninteracting Majorana fermion hopping in a fixed background $\mathbb{Z}_2$ gauge field. The gauge-invariant content of the background is in the $\mathbb{Z}_2$ plaquette fluxes, given by the product $\prod_{\langle ij \rangle \in \partial p} u_{ij}$ counterclockwise around p, and is equal to $-W_p$. The ground state of H is that which minimizes the fermionic energy among

[1] On a torus, one has $\prod_p W_p = 1$. There are also two independent commuting loop operators extending across the cycles of the torus, $V_x = \prod_i X_i$ along a loop of alternating YY' and ZZ' bonds, and $V_y = \prod_i Y_i$ along a loop of alternating XX' and ZZ' bonds. Both $V_{x,y}$ commute with H and all the W_p.

$$H = J_x {\sum_{\langle ij\rangle}}' X_i X_j + J_y {\sum_{\langle ij\rangle}}' Y_i Y_j + J_z {\sum_{\langle ij\rangle}}' Z_i Z_j$$

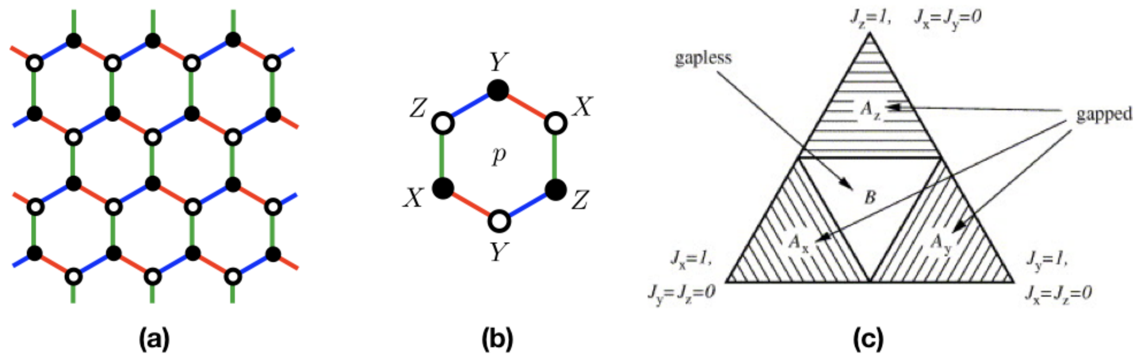

Figure 1: The Kitaev honeycomb model: (a) The arrangement of bond interactions. (b) The plaquette stabilizer W_p is a product of spin operators around hexagon p. (c) The phase diagram in the $J_x + J_y + J_z = 1$ plane exhibits gapped and gapless $\mathbb{Z}_2$ spin liquid phases.

all the 2^{N_c+1} gauge-inequivalent backgrounds, where N_c is the number of hexagonal plaquettes[2]; the number of lattice sites is $N = 2N_c$. The phase diagram in the $J_x + J_y + J_z = 1$ plane exhibits gapped and gapless $\mathbb{Z}_2$ quantum spin liquid (QSL) regimes[3].

Recently, two groups have investigated the dynamics of $S = \frac{1}{2}$ spins on the honeycomb lattice subjected to repeated measurements [2, 3]. Their protocol is as follows. First initialize the system into a maximally mixed density matrix $\rho(0) = 2^{-N}\mathbb{I}$, corresponding to an infinite temperature state. At each subsequent micro time step, choose with probability p_α a random bond of type α and perform a projective measurement of the two-qubit operator $\sigma_i^\alpha \sigma_j^\alpha$ on that bond. The time t is defined in units of full sweeps, *i.e.* N consecutive random projective bond operator measurements.

The iterated projective measurements of the monitored Kitaev spin liquid (MKSL) are one example of a broad class of monitored quantum dynamics [4], which studies properties, such as entanglement, of the trajectories taken by quantum many-body systems. A projector $\sqcap$ satisfies $\sqcap = \sqcap^2 = \sqcap^\dagger$, and has eigenvalues 0 or 1. The probability that $\sqcap = 1$ is given by the Born rule, $\mathsf{Prob}(\sqcap = 1 \,|\, \rho) = \mathsf{Tr}(\sqcap\, \rho)$. After measurement $\mathcal{M}$, the density matrix is given by [5]

$$\mathcal{M} \,:\, \rho \mapsto \mathcal{M}(\sqcap \,|\, \rho) = \frac{\sqcap\, \rho\, \sqcap}{\mathsf{Tr}(\sqcap\, \rho)} \quad . \tag{2}$$

Measurement can affect entanglement properties, as the following two trivial examples confirm. First, starting with any N-spin density matrix, perform a sequence of single spin measurements on each of the sites. This collapses the initial state to a product state, and

[2] This accounts for $N_c - 1$ independent plaquettes, plus the two independent Wilson loops on the torus. When the lattice has reflection symmetry in a line which does not intersect any lattice sites, the job of finding the minimizing flux background is greatly simplified by a theorem due to E. Lieb.

[3] Breaking time reversal symmetry in the gapless phase opens a gap and results in nonabelian excitations corresponding to Majorana fermions bound to $\mathbb{Z}_2$ plaquette flux vortices.

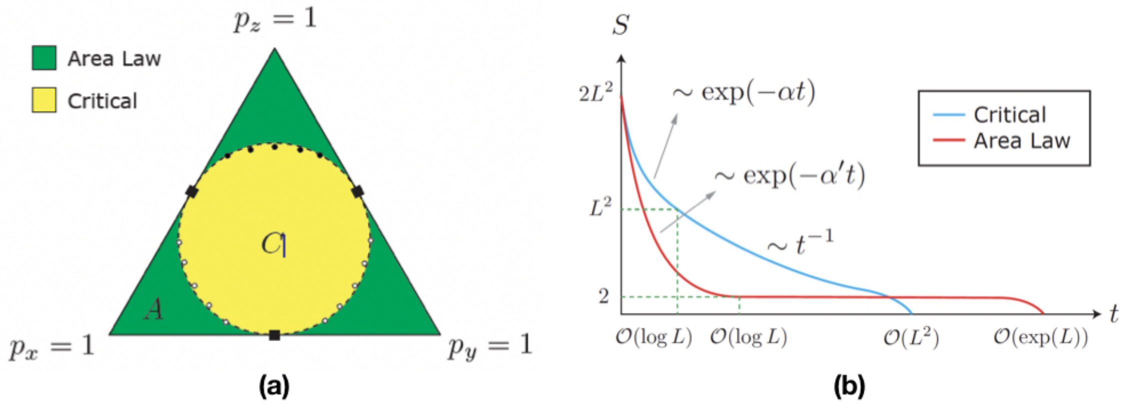

Figure 2: Monitored Kitaev spin liquid: (a) Steady state phase diagram. (b) Purification dynamics and behavior of the entropy S. For the blue curve, the initial state is mixed, but with a frozen flux configuration. The number of plaquettes is L^2. From ref. [2].

any initial entanglement entropy is completely lost. Second, starting with the pure product state $|\,\Psi\,\rangle = \prod_n |\!\uparrow\rangle_n$, perform a sequence of measurements of the products $X_{2j}X_{2j+1}$ on half of the bonds. This has the effect of replacing $|\!\uparrow\uparrow\rangle \to 2^{-1/2}\big(|\!\uparrow\uparrow\rangle \pm |\!\downarrow\downarrow\rangle\big)$ on each of the measured bonds, where the sign corresponds to the measurement outcome. Thus, the initial unentangled product state is replaced by $\frac{1}{2}N$ singlets. Two special features of the MKSL protocol are essential to their monitored dynamics and steady state spin liquid properties: (i) the bond operators which share a common site do not commute with each other, and (ii) each bond operator commutes with all the plaquette operators W_p.

While the Hilbert space of the Majorana partons at each site is four dimensional, and one must apply the projector $P = \prod_i \frac{1}{2}\big(1 + c_i b_i^x b_i^y b_i^z\big)$ to the Majorana degrees of freedom in order to obtain the (two-dimensional) Hilbert space of spins, all spin operators and products thereof (*e.g.*, the bond operators $\sigma_i^\alpha \sigma_j^\alpha$) commute with P. This allows the analysis the parton dynamics of the unprojected density matrix $\rho_f(t)$ in the enlarged Hilbert space [2].

Two remarkable things happen under the MKSL protocol. First, the density matrix, when expressed in terms of the Majorana partons, *purifies*, and on a time scale t^* approaches the form $\rho_f(t) = \rho_c(t) \otimes \rho_b$, where $\rho_b = |\,\Psi_b\,\rangle\langle\,\Psi_b\,|$ is a pure state [2]. The subsequent entanglement dynamics are described in terms of the c partons. Second, as a function of the probabilities $\{p_x, p_y, p_z\}$ there are two exotic steady-state phases (fig. 2a): (i) a topologically ordered phase, in which the monitored pure states exhibit topological order, related to that of the Kitaev toric code, as well as area law entanglement and quantized topological entanglement entropy $S_{\rm top} = -\log 2$, and (ii) a critical phase, in which the entanglement entropy of a subsystem A scales as $S_A = c_0\, L_A \log L_A + \ldots$, with c_0 a nonuniversal constant.

The purified b-parton density matrix takes the form $|\,\Psi_b\,\rangle\langle\,\Psi_b\,| = \prod_p \frac{1}{2}(1 \pm W_p)$, corresponding to a frozen configuration of plaquette $\mathbb{Z}_2$ fluxes for $t > t^*$. In the Hamiltonian Kitaev model, different flux configurations result in changes in the interference properties of the hopping c-partons, leading to changes in the energy spectrum. For the monitored dynamics, however, the distribution of these plaquette fluxes does not much affect the sub-

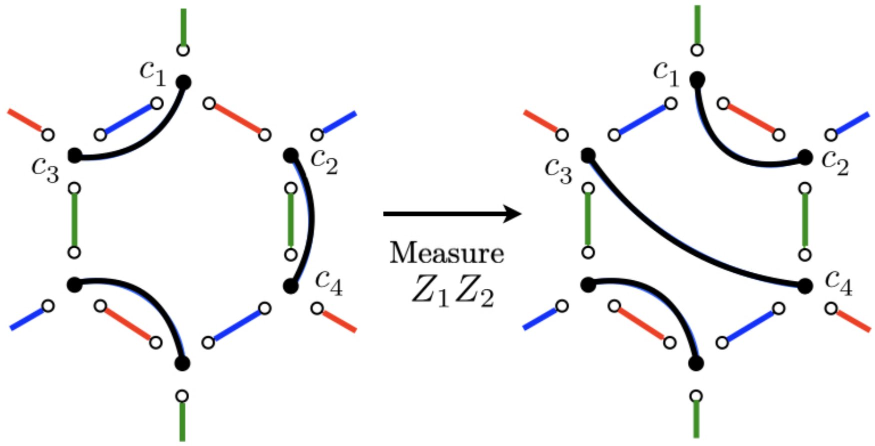

Figure 3: A projective measurement of a bond spin operator acting on a state $|\,\Psi_c\,\rangle$ in which the c-partons are paired into complex fermions results in a pairing reassignment. Adapted from ref. [2].

sequent c-parton entanglement dynamics for $t > t^*$, which are illuminated by the following considerations. A single localized fermionic orbital has a two-dimensional Hilbert space. For Majorana fermions, this degree of freedom is distributed nonlocally: one can form a complex fermion $\chi = \frac{1}{2}(c_k + ic_l)$ from Majoranas on two different sites k and l. One can loosely say that the Hilbert space for a single Majorana is of dimension $\sqrt{2}$. By pairing up the N Majoranas, one obtains $2^{N/2}$ states, and multiplying this by the $2^{N/2}$ plaquette flux configurations and one recovers the 2^N dimensional Hilbert space of the original spins. Starting with a state $|\,\Psi_c\,\rangle$ in which the c-partons are all paired up, measurement of a bond operator results in a reassigned pairing, as depicted in fig, 3. Explicitly, starting from a state $|\,\psi\,\rangle$ in which $ic_1c_3 = ic_2c_4 = ib_1^zb_2^z = 1$, we apply the bond projector $\frac{1}{2}(1 + Z_1Z_2) = \frac{1}{2}(1 + c_1c_2b_1^zb_2^z)$ and obtain a state with $ic_2c_1 = ic_3c_4 = ib_1^zb_2^z = 1$, *viz.*

$$
\begin{aligned}
|\,\psi'\,\rangle &= \bigg(\frac{1 + c_1c_2b_1^zb_2^z}{2}\bigg)\overbrace{\bigg(\frac{1 + ic_1c_3}{2}\bigg)\bigg(\frac{1 + ic_2c_4}{2}\bigg)\bigg(\frac{1 + ib_1^zb_2^z}{2}\bigg)|\,\psi_0\,\rangle}^{|\,\psi\,\rangle} \\
&= \bigg(\frac{1 + ic_2c_1}{2}\bigg)\bigg(\frac{1 + ic_3c_4}{2}\bigg)\bigg(\frac{1 + ib_1^zb_2^z}{2}\bigg)|\,\psi_0'\,\rangle \quad ,
\end{aligned}
\tag{3}
$$

where $|\,\psi_0'\,\rangle = \frac{1}{2}(1 + ic_1c_3)|\,\psi_0\,\rangle$.

In the initial maximally mixed state, the von Neumann entropy $S = -\mathsf{Tr}(\rho\log\rho)$ is $S(0) = N\log 2$. In the topologically ordered phase, as shown in fig. 2b, $S(t)$ drops rapidly to $S(t^*) = 2\log 2$ on a time scale $t^* \sim \mathcal{O}(\log L)$, where L is the linear dimension. This value is associated with the information contained in the two independent topologically nontrivial Wilson loops wrapping around the torus. Ultimately, on a time scale $t_{\text{purif}} \sim \mathcal{O}(\exp L)$, the system completely purifies. Thus, two logical qubits are protected for an exponentially long time in the system size; these qubits are also encoded in the Hamiltonian ground state. The area-law entanglement entropy in this phase is also found in the gapped $\mathbb{Z}_2$ spin liquid phases of the Hamiltonian ground state, and in general the properties of the steady state monitored phase are quite similar to those of the gapped $\mathbb{Z}_2$ Hamiltonian QSL phase.

In the critical phase, and starting with a mixed state with a frozen flux configuration, the von Neumann entropy drops as a power $S(t) \sim L^2/t$, and ultimately purifies[4]. The entanglement entropy behaves as $L_A \log L_A$, similar to what is found in systems with a Fermi surface, whereas the Hamiltonian ground state of the gapless phase is a semimetal. In this context, note that the random application of bond projectors destroys the translational symmetry which, along with time-reversal symmetry, prohibits a Fermi surface in the ground state [2, 6]. The phase transition between the two phases of the monitored Kitaev spin liquid is argued [2] to be described by a geometrical phase transition of an associated three-dimensional classical loop model [7], and is in the universality class of the 3D Anderson localization transition of symmetry class C.

The advent of 'synthetic quantum matter' in quantum simulators created from trapped ions, Rydberg atom systems, and superconducting qubits, has opened a window onto many potentially new and exotic quantum phases of matter. Conjuring steady state topological phases from some convenient initial state via a protocol consisting exclusively of measurements suggests that monitored dynamics may be a tool for generating exotic quantum phases which might potentially have applications for quantum information storage.

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References

[1] A. Kitaev, *Ann. Phys.* **321**, 2 (2006).

[2] A. Lavasani, Z.-X. Luo, and S. Vijay, arXiv:2207.02877.

[3] A. Sriram, T. Rakovsky, V. Khemani, and M. Ippoliti, arXiv:2207.07096.

[4] See §4 of M. P. A. Fisher, V. Khemani, A. Nahum, and S. Vijay, arXiv:2207.14280.

[5] K. Hornberger, Introduction to Decoherence Theory, in *Entanglement and Decoherence*, A. Buchleitner, C. Viviescas, M. Tiersch eds. (Springer, 2009).

[6] For an example of a Kitaev chiral spin liquid with a stable Fermi surface on the kagome lattice, see V. Chua, H. Yao and G. A. Fiete, *Phys. Rev. B* **83**, 180412(R) (2011). It might be interesting to explore the monitored dynamics protocol within the context of this model.

[7] E. J. Beamond, J. Cardy, J. T. Chalker, *Phys. Rev. B* **65**, 65 (2002).

[4] This was shown in ref. [2] by appealing to the 3D loop model description. Starting from an infinite temperature density matrix $\rho(0) \propto \mathbb{I}$, the purification dynamics of $S(t)$ may be different [3].