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Whither many-body localization?

Quantum chaos challenges many-body localization.
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Chaos and ergodicity are two closely related cornerstones of physics. In classical systems chaos is usually defined through the instability of the system's trajectories to either small changes in initial conditions or small changes of the Hamiltonian.Generally chaos is believed to destroy all conservation laws except energy and perhaps a few more like the particle number. Then a standard textbook argument [1] tells us that the time averaged probability distribution of a chaotic isolated system depends only on these few conserved quantities, leading to statistical equilibrium. In Fig. 1 we illustrate how this reasoning works for a single-particle of a unit mass in a two-dimensional nonlinear potential $V(x,y) = (x^2+y^2)/2+2x^2y^2$. Two trajectories with slightly different initial conditions exponentially separate from each other at short times (left panel) indicating chaos and fill the allowed phase space indicating ergodicity.

While generic systems of interacting particles are believed to be ergodic in the thermodynamic limit, this is not always the case in few-particle systems. A powerful theorem due to Kolmogorov, Arnold and Moser (KAM) [2] tells us that there can be a finite chaos threshold required to break all the conservation laws. In particular, if the unperturbed integrable system has a conserved quantity q_0 then after adding a small integrability breaking perturbation ϵ there can be a deformed conserved charge

$$Q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \dots \tag{1}$$

As long as the expansion (1) converges there is an extra conservation law constraining dyanmics and thus preventing the system from thermalization. These deformed integrals of motion Q are often called LIOMs (local integrals of motion). The expansion (1) also implies that the transition from integrable to chaotic behavior is necessarily non-perturbative.

In quantum systems there is still no consensus on how one should define chaos and ergodicity. Now both concepts are most commonly understood through the eigenstate thermalization hypothesis (ETH) (see Re. [3] for the review), which is based on the random matrix theory. Like in classical systems emergence of LIOMs in the form of Eq. (1), where q_j stand for local Hermitian operators, is a sufficient condition for violating ergodicity.

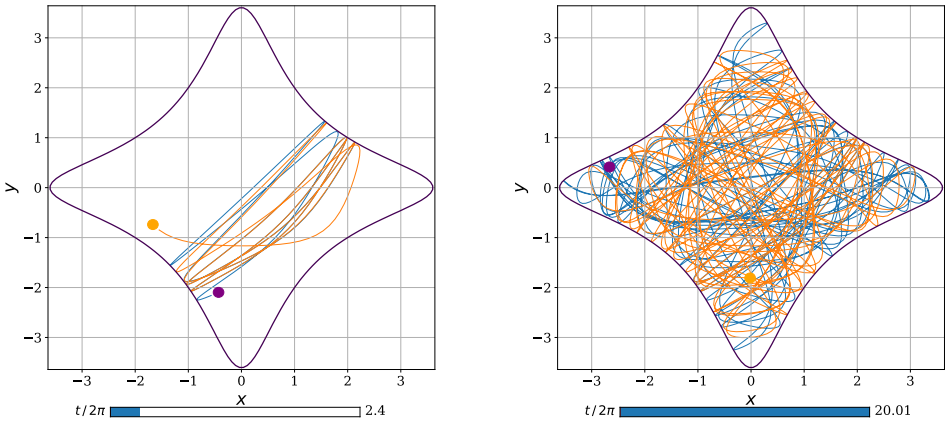

Figure 1: Two trajectories of a particle with slightly different initial conditions in a nonlinear potential. Left and right panels correspond to short and long times, respectively.

Alleged existence of such dressed LIOMs, perturbatively connected to non-interacting localized Anderson orbitals, is the foundation for existence of the many-body localized (MBL) phase [4, 5]. Because the many-particle Hilbert space is exponentially large in the system size, this LIOM stability was initially coined Fock space localization. Each site in this Fock space represents a particular configuration of charges on the lattice sites. In Fig. 2 we schematically illustrate this idea, where the left panel corresponds to ergodic, delocalized phase and the right panel would correspond to the localized MBL phase.

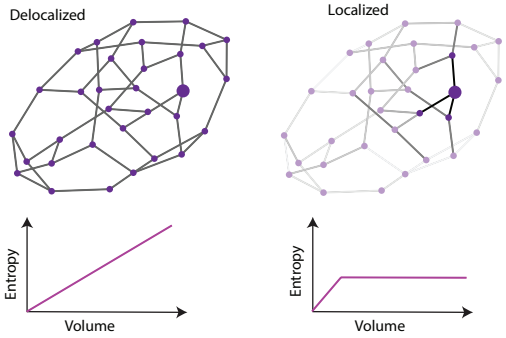

Figure 2: A schematic illustration of MBL as a localization in the Fock space.

Most numerical studies focused on one-dimensional systems of spinless fermions, which can be mapped via the Jordan-Wigner transformation [6] to spin 1/2 chains with a disordered magnetic field. The most extensively studied model, is the disordered XXZ chain, described by the Hamiltonian:

$$H_{\rm XXZ} = \sum_j (S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta S^z_j S^z_{j+1}) + \sum_j h_j S^z_j, , \qquad (2)$$

Here the first two terms represent the hopping of fermions, which is equivalent to the transport of the z-magnetization in the spin language. The term $\sum_j \Delta S^z_j S^z_{j+1}$ encodes interactions between the fermions or spins and finally the last term represents the disordered potential/magnetic field. Here we set the hopping strength to unity. Δ is typically chosen to be of the order of unity ($\Delta = 1$ corresponds to the Heisenberg chain) and h_j are typically chosen to be independently and uniformly distributed on each site in the interval $[-W, W]$. Many early numerical studies concluded that there is a transition in this model to the localized phase happening around disorder $W_c \approx 3.5 - 3.7$ (see Ref. [7] for review). However, those

early numerical results were challenged by a growing number of papers using improved numerical methods [8, 9, 10, 11] pushing this bound to much higher value $W_c \gtrsim 20$. Combining this observation with very strong (exponential) dependence of the relaxation time on disorder in the ergodic phase it becomes obvious that in order to distinguish localization from slow relaxation one should wait for astronomically long times, which are not accessible in experiments. In Ref. [12] and the follow up work [13] J. Šuntajs et. al. suggested for the first time that numerical results are consistent with an extensive scaling of the critical disorder strength with the system size, i.e. with absence of localization in the thermodynamic limit.

Early analytical papers on MBL [4, 5] argued about the stability of LIOMs by doing perturbation theory around the non-interacting localized problem. Later in Refs. [14, 15] it was proposed that the MBL phase in spin systems is defined through existence of a convergent expansions like (1), where the role of q_0 is played by the local z-magnetizations S^z_j, $j = 1, 2, \dots L$, i.e. there is a quasi-local unitary transformation that deforms the local magnetization into a LIOM. It was argued that MBL is very similar to single-particle localzation on a Cayley tree or even better random regular graphs (RRGs) [16, 17], which are locally tree like. However, in order to make the analogy with trees work one needs to introduce extra assumptions that the connectivity of these graphs does not scale with volume of the system as one would naively expect. In addition, mapping MBL to the Fock space localization on a finite degree Cayley tree would imply system size independent participation entropy of localized eigenstates in Fock space, see Fig. 2. This, however, would contradict the fact that any small local basis transformation, which need not delocalize the system in physical space, will generate extensive entropy in Fock space. One should thus at best hope for MBL to be transition from fully ergodic to a system with finite fractal dimension, which cannot be described by Anderson localization in Fock space.

The arguments based on convergence of the perturbative expansion, and the mapping of interacting disordered models to RRGs, are also largely insensitive to dimensionality of the system. A different line of arguments, which again explicitly uses the many-body and local nature of the problem, suggested that this picture cannot be correct and localization can only be stable in one dimension. The key insight, known as avalanche instability, was proposed in Ref. [18]. It was argued that a single ergodic island in a system, coming from a rare low disorder configuration, will melt the whole localized phase in dimensions higher than one. In one dimension this argument implies that the MBL can only be stable if the correlation length is less than the lattice spacing, which is again in direct contradiction with both the pioneering MBL papers and the Cayley tree arguments. The avalanche argument seems to be very powerful as it shows self consistency of a localized phase: if the correlation length of LIOMs is short enough then they are stable against thermal inclusions. Conversely, if the localized phase does exist, it must be stable against such inclusions and hence the localization length must be sufficiently short. However, self consistency is a necessary condition for validity of a certain logical statement, but it is never sufficient. The key question missing in this line of thought is whether the localization length exists at all, i.e. whether LIOMs do have exponential tails. In fact, existence of these tails was a key unjustified assumption, which was the weak point of the whole MBL construction.

To understand the microscopic structure of the LIOMs and the mechanism of the flow of the correlation length with the distance one can consider a simplified setup where a spin with a very strong magnetic field is weakly coupled to either a clean or disordered bath

(see Fig. 3). This setup corresponds to the Hamiltonian $H + VS_0^z + \epsilon H_{\rm int}$. Then Ref. [19] showed that one can develop an expansion for the LIOM like in Eq. (1) with $q_0 = S_0^z$. If we stop in the n-th order, in $1/V$ and the first order in ϵ the decay rate of this LIOM is given by:

$$\Gamma_n^2 \approx \epsilon^2 \frac{\|Ad_H^{2n+1} H_{\rm int}\|^2}{V^{4n+2}}. \tag{3}$$

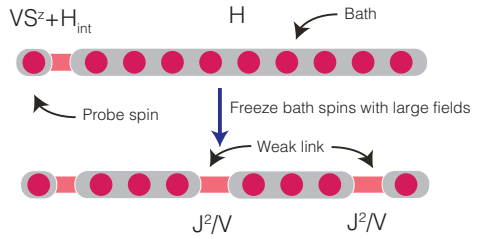

Figure 3: LIOM construction setup.

Convergence of the LIOM in the leading order in ϵ is thus tied to the scaling of the norm of the nested commutator of $H_{\rm int}$ with H. Such scaling was recently studied in the literature following Ref. [20]. For non-interacting models it is exponential: $\|Ad_H^n H_{\rm int}\| \sim \kappa^n$ and the LIOM construction converges for $V > \kappa$. A similar situation applies to RRGs. However, in local interacting models these norms have factorial scaling: $\|Ad_H^n H_{\rm int}\|^2 \sim n!\,\kappa^n$ with additional logarithmic corrections in one dimension [21]. In this way the LIOM construction is asymptotic and there is an optimal order $n^* \sim V/\kappa$ beyond which the decay rate of the LIOM stops decreasing with n. It is interesting that this conclusion does not depend on whether the Hamiltonian H is disordered or not. Indeed in the bottom plot of Fig. 3 it is illustrated how other strongly disordered sites in the system can be replaced by weak links connecting the spins on the left and right of these sites via a virtual process. Such weak links can only lead to a finite renormalization of the optimal order n^*, but cannot fight the factorial growth of the nested commutators. The origin of the LIOM instability is thus the rapidly growing number of the virtual transitions with each order of perturbation theory rather than divergence of individual perturbative contributions with smallest denominators. Interestingly one arrives to the same conclusion by going to the rotating frame with respect to VS_0^z and applying Fermi's Golden Rule (FGR) to the rapidly oscillating coupling $H_{\rm int}$. When the FGR rate becomes comparable to the level spacing the LIOM cannot decay leading to the relation between the system size and the potential strength describing the crossover between the localized and delocalized regimes: $L^* \sim V/\kappa$ exactly as first predicted in Ref. [12].

Since the order n of the expansion is tied to the spatial extent of the LIOM, one can interpret the scaling of Γ_n as the growth of the LIOM localization length with distance: $\xi^{-1}(x) \sim (\log V - \log x)$. The fact that it decays slower and slower with n means that the LIOM spreads more and more as its support increases. This conclusion agrees with the scaling of the slowest operators with the system size [9, 10]. This situation is schematically illustrated in Fig. 4, where the real LIOM (red line) becomes unstable after reaching a finite size and then decays to the continuum. In this sense the LIOM is similar to a dressed quasi-particle, with a long but finite life time.

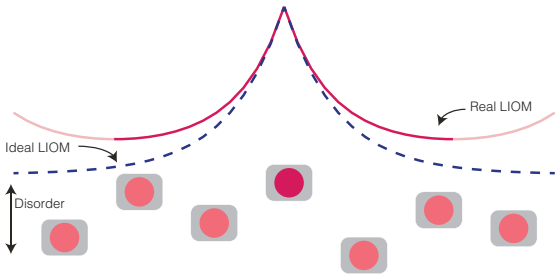

Figure 4: A schematic representation of a presumed idealized LIOM with exponential tails and the real unstable LIOM.

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