Journal Club for Condensed Matter Physics
https://www.condmatjclub.org
DOI:10.36471/JCCM_January_2023_02

Sharpening the Conformal Symmetry of the 3D Ising Transition by Fuzzy Sphere Simulations

Uncovering conformal symmetry in the 3D Ising transition: State-operator correspondence from a fuzzy sphere regularization

Authors: Wei Zhu, Chao Han, Emilie Huffman, Johannes S. Hofmann, Yin-Chen He
arXiv:2210.13482

Recommended with a Commentary by Ashvin Vishwanath, Harvard University

Symmetry enhancement is one of the most commonly encountered, but also one of the most remarkable examples of emergence. Many systems tuned to critical points, both quantum and classical, effectively display much more symmetry in the limit of low energies and large length scales, as compared to the microscopic models from which they descend. For instance the critical point in the 3D Ising model, an iconic model of classical statistical mechanics, describes the ordering transition of a 3D ferromagnet (with spin anisotropy) on cooling below the Curie temperature $T = T_C$. Besides Ising symmetry, the model only has some discrete translation and rotation symmetry due to the lattice. However at the critical point, the physics at long length scales is believed to acquire not just the full translation and rotation symmetry of uniform space, but is also invariant under other spatial transformations that preserve shapes (the conformal transformations). These conformal transformations include global scale transformations or dilations, $\vec{r} \to (1+\epsilon)\vec{r}$ but interestingly also include an additional set of special conformal transformations*.

The well known scaling laws and exponents at critical points reflect only a small part of the symmetry enlargement. The featured reference uses numerical calculations on relatively small sized systems to convincingly establish the full conformal symmetry of the 3D Ising critical point, and obtain a great deal of information about it beyond critical exponents. The actual model that is studied is a 2+1D quantum model, related to the classical 3D Ising model by the standard quantum - classical mapping. What is non-standard though is the definition of the model on a *sphere* that unlocks a one to one correspondence between energy eigenstates and the 'scaling operators' that transform in a simple way under conformal mappings. The energy eigenvalues then map to the scaling dimensions of the operators. Previous attempts to define the Ising model on the sphere had to grapple with introducing a short distance cutoff - usually done in the form of a lattice - which then requires the

*Which take the form $r_i \to r_i + 2(\vec{\epsilon}\cdot\vec{r})r_i - \epsilon_i r^2$. Note that if we were in 2D the set of conformal transformations would be even larger, owing to the fact that holomorphic functions $f(z)$ generate conformal maps. Here we are interested in 3D classical models, or equivalently 2+1D quantum models with effective Lorentz symmetry.

introduction of lattice defects to accommodate the curved surface of the sphere. In the featured reference, this problem is sidestepped by using a different regularization scheme - that of charged particles on a sphere in a magnetic field. Such a 'fuzzy sphere' setup which is described in more detail below, provides a different and more symmetric way to define a finite model. Most surprisingly, this new regularization appears to give accurate results for scaling dimensions even at very small system sizes (essentially involving just 16 spins!). The accuracy of the results can be verified by comparing it to an entirely different numerical approach, the 'conformal bootstrap' (CB) [1], which proceeds by imposing a large number of consistency constraints that follow from conformal invariance, to severely narrow down allowed values of scaling dimensions. Comparing results from the featured reference to CB, excellent agreement is found for the scaling dimension of about 70 (!) scaling operators with angular momentum $l \leq 4$, whose scaling dimensions are found to be within a few percent of the CB values. Additionally, the scaling dimensions of two primary operators that were not accessible to CB were also obtained. Before describing the results of the featured reference in more detail, let us recall some relevant background.

Background: Scale invariance at critical points is typically discussed in terms of critical exponents. For instance, the associated divergence of the intrinsic length scale ξ as T_C is approached defines ν via $\xi \sim |T - T_C|^{-\nu}$, and similarly the spontaneous magnetization in the ordered phase for $T \leq T_C$ is controlled by β via $m \propto (T_C - T)^{\beta}$. From the renormalization group point of view, this corresponds to two relevant operators, ϵ and σ which correspond to tuning to the critical temperature and to zero field, whose scaling dimensions, i.e. their behavior under dilations, are related by: $\Delta_\epsilon = 3 - \nu^{-1}$ and $\Delta_\sigma = \beta\nu^{-1}$. Using different schemes including Monte Carlo numerics, field theory and experiment, these have been accurately obtained. However these are just the tip of the iceberg, being the few scaling operators with the lowest dimensions. In general, a conformally invariant theory in 3D contains an infinite set of primary operators ϕ_i with dimension Δ_i and spin l_i along with coefficients C_{ijk}. All other scaling operators can be derived from one of the ϕ_i.

How does one concretely relate a scaling field to microscopic degrees of freedom? One route uses the powerful state-operator correspondence that is rooted in conformal invariance, which allows local operators to be related to states. At first sight this may seem like a 'type' mismatch, as well as a mismatch in dimension, local operators being points in D dimensions, while quantum states live on a $D-1$ dimensional equal-time slice. However, the transformation in the Figure 1A illustrates how this can be achieved, which maps the radial coordinate on the left 'r' to a time coordinate 't' on the right via: $t = \log r$. Thus the effect of dilation $r \to (1+\epsilon)r$ on the left, is simply time evolution on the right $t \to t+\epsilon$ produced by the Hamiltonian. The eigenvalues of the time evolution, i.e. the energies, are then nothing but scaling dimensions, and eigenstates are related to scaling operators. Note that the time slice on the right corresponds to a sphere i.e. S^{D-1}. In $D = 2$, where this is just a circle, the relation between eigenstates and scaling operators has been heavily exploited in the past. However, here in $D = 3$, we need to define our system on the two dimensional surface of a sphere S^2. In contrast, the simplest periodic boundary conditions geometry leads to a *torus* for which this state operator correspondence does not directly hold. Hence, one is confronted with the question of placing a CFT on a sphere while maintaining a finite Hilbert space, to enable a direct numerical calculation.

The Fuzzy Sphere: The creative solution to this problem in the featured reference

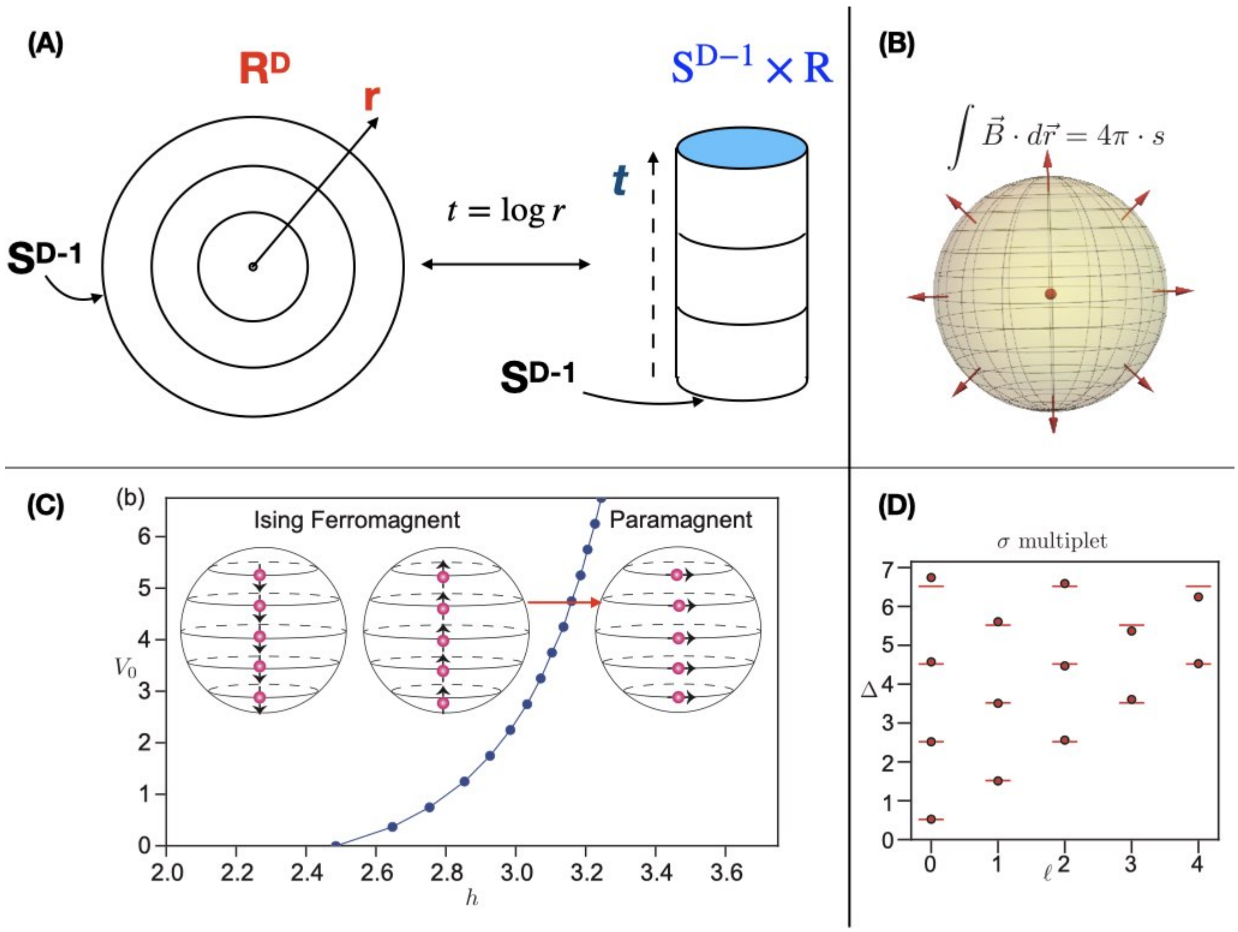

Figure 1: (A) State operator correspondence relating dilations in D-dimensional statistical mechanics model to time evolution of a quantum state on the sphere S^{D-1}. (B) The fuzzy sphere setup, electrons on a sphere in a magnetic field restricted to the lowest Landau level gives a finite dimensional Hilbert space. (C) A magnetic phase transition on the fuzzy sphere in the Ising universality class is tuned by a field h and the energy spectrum at the transition point is examined. (D) An example of part of the energy spectrum. The energies correspond to scaling dimensions, and the lowest energy state is the primary operator σ with $\Delta_\sigma = 0.524$, close to the best existing estimates.

exploits the fact that charged particles in a magnetic field give rise to degenerate Landau levels. The degeneracy of the lowest Landau level grows with area, and hence for a finite surface like a sphere, one automatically has a finite Hilbert space. However, unlike a lattice based regularization, one retains the rotation symmetries of the sphere, which allows for a labeling of states by angular momenta and also seems to mitigate finite size effects. The precise model involved taking $N = 16$ interacting electrons on a sphere with the same number of available states. If the electrons are spin polarized, a unique fully filled shell is obtained. However, if the electrons have to spontaneously pick a spin direction, this corresponds to a ferromagnet. By appropriately designing interactions between electrons the phase diagram shown in Figure 1C was obtained. A spin anisotropy term favors an Ising alignment of electron spin along the S_z direction, while a Zeeman field in the S_x direction triggers a phase transition in the 2+1D Ising universality class. The system is tuned to sit on the phase boundary, and its energy spectrum is calculated using exact diagonalization.

Results: The energy spectrum on the fuzzy sphere for $N = 16$ was obtained in the featured reference, along with the corresponding angular momentum l of several excited states. Relating energy to the scaling dimension requires calibration with an operator with known scaling dimension (here, the stress-energy tensor which has $l = 2$ and $\Delta_T = 3$). The scaling dimensions of other operators can then be read off, one family is shown as an example in Figure 1D. Accurate scaling dimensions for 11 other primary operators were obtained which agree within 1.6% of CB values . Notably, in contrast to many other simulations of critical phenomena these calculations are performed at a fixed size.

In the future, a better understanding of finite size effects in this setup is clearly worth pursuing. For instance, for reasons that remain fuzzy, finite size effects vary along the phase transition line. The fuzzy sphere geometry also allows for the exploration of the 3D analog of the 2D central charge, denoted as 'f' and the analogous f-theorem which constrains 3D criticality [2]. Finally this technique should be deployed on more exotic critical points - for instance the deconfined phase transitions - models for which were already discussed in the quantum Hall context [3].

In closing it is worth noting that the first experiments on the critical point in liquid-gas transitions (which shares the 3D Ising universality class) by Charles Cagniard de la Tour , took place 200 years ago ! It is incredible that its physics continues to surprise us.

I thank Ruihua Fan and Shai Chester for useful discussions and comments.

References

[1] D. Poland, S. Rychkov and A. Vichi, *The Conformal Bootstrap: Theory, Numerical Techniques, and Applications*, *Rev. Mod. Phys.* **91** (2019) 015002 [1805.04405].

[2] Quantum Entanglement sheds new light on an old problem. Posted in Journal Articles on January 31, 2014 by JCCMP. Link

[3] M. Ippoliti, R. S. K. Mong, F. F. Assaad and M. P. Zaletel, *Half-filled Landau levels: A continuum and sign-free regularization for three-dimensional quantum critical points*, *prb* **98** (2018) 235108 [1810.00009].