Many body quantum games and phases of matter

1. **A multi-player, multi-team nonlocal game for the toric code**
   Authors: Vir B. Bulchandani, Fiona J. Burnell, and S.L. Sondhi
   Phys. Rev. B 107, 035409 (2023)

2. **Playing nonlocal games with phases of matter**
   Authors: Vir B. Bulchandani, Fiona Burnell, and S.L. Sondhi
   Phys. Rev. B 107, 045412 (2023)

Recommended with a Commentary by Rahul Nandkishore,
Department of Physics and Center for Theory of Quantum Matter,
University of Colorado Boulder

Quantum phases of matter are conventionally characterized in terms of their (local and non-local) correlation functions. The advent of quantum information processing architectures, however, prompts one to wonder if the correlations inherent in phases of matter could be harnessed to gain quantum advantage in some tasks. This provides an alternative way of thinking about correlations and phases, complementary to the traditional condensed matter perspective, but intimately related to the perspective on generalized Bell tests developed by Mermin. A crisp example is provided by the 'parity game' of Brassard, Broadbent and Tapp.

We begin by discussing the parity game with three players: Alice (A), Bob (B), and Charlie (C), who are not allowed to communicate with each other during the course of the game. The three players receive input bits a, b and c respectively, with $c = (a + b) \bmod 2$. Each player is asked to output a bit y such that $(y_A + y_B + y_C) \bmod 2 = \frac{a+b+c}{2} \bmod 2$. If the output bits satisfy this condition, then the players 'win' the game, else they lose. Now, suppose that Alice follows a deterministic strategy whereby for any input a she outputs $y_A(a)$, and similarly for Bob and Charlie. 'Winning' the game requires that

$$y_A(0) + y_B(1) + y_C(1) = 1 \tag{1a}$$

$$y_A(1) + y_B(0) + y_C(1) = 1 \tag{1b}$$

$$y_A(1) + y_B(1) + y_C(0) = 1 \tag{1c}$$

$$y_A(0) + y_B(0) + y_C(0) = 0, \tag{1d}$$

where the additions on the left hand side are all mod 2. However, these equations cannot all be satisfied! To see this, note that adding all four equations mod 2 produces the contradiction $0 = 1$. The best that can be achieved in the absence of communication between players is to satisfy three out of four equations. Accordingly, the best that can be achieved by any classical strategy is to win the game 3/4 of the time (this bound may be saturated by e.g. the strategy 'always output 1').

Now, suppose A, B, and C share a Greenberger, Horne and Zeilinger (GHZ) state $|GHZ^+\rangle_3 = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ with each player holding one qubit. In this case, they can win the game with probability one, as follows: if a player receives the input bit 0 then they measure their qubit in the X basis (i.e. measure the X spin component of their qubit); if they receive input bit 1, then they instead measure in the Y basis (i.e. measure the Y spin component). In either case, they output $y_i = (1 - \lambda_i)/2$ given the measurement outcome $\lambda_i \in \{1, -1\}$ from their local measurement. It may be readily verified that this strategy succeeds for every choice of input bits. More generally, if the players share a resource state $|\psi\rangle$, then this strategy succeeds with probability

$$p_{\mathrm{q}}(\psi) = \frac{1}{2} + \frac{1}{8} \langle\psi|M_3|\psi\rangle \ . \tag{2}$$

where $M_3 = X_1X_2X_3 - X_1Y_2Y_3 - Y_1X_2Y_3 - Y_1Y_2X_3$ is called a Mermin polynomial, the X and Y are Pauli operators, and ${}_3\langle GHZ^+|M_3|GHZ^+\rangle_3 = 4$.

This game generalizes to an arbitrary number of non-communicating players N, where each player is given an input bit $a_j \in \{0, 1\}$, with the promise that $\sum_{j=1}^{N} a_j$ is even. Each player is asked to output a classical bit $y_j \in \{0, 1\}$ satisfying

$$\left(\sum_{j=1}^{N} y_j\right) \bmod 2 = \frac{\sum_j a_j}{2} \bmod 2 \tag{3}$$

For $N = 3$ this is the same win condition that we discussed before. However, in the large N limit the 'win condition' is equally likely to require output bit strings of even or odd parity; accordingly there is no classical strategy that 'wins' with probability greater than $1/2$ in the large N limit. For arbitrary N it was shown by Brassard, Broadbent and Tapp that the optimal classical strategy wins with probability $\frac{1}{2} + \frac{1}{2^{\lceil N/2 \rceil}}$, where $\lceil\rceil$ denotes the ceiling function (for $N = 3$ this reproduces the classical limit of $3/4$ win probability discussed above). However, just as in the $N = 3$ case discussed above, victory can be guaranteed for any N if the players are provided with a shared resource state $|GHZ^+\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$. One can show that if each player measures in a basis dependent on her input bit a_i, and outputs the measurement result, that the win condition (3) will be satisfied with probability one. This should be understood as a many-player generalization of Bell tests - the quantum correlations inherent in an entangled state can be exploited by non-communicating players to output answers that are more correlated than any classical strategy would permit. That is, the state $|GHZ^+\rangle$ serves as a resource state providing quantum advantage at the task of winning the parity game.

The authors of the highlighted articles take this observation as a jumping off point to investigate the ability of correlations inherent in ordered states of quantum matter to provide quantum advantage more generally. In the first of the highlighted papers (1), they invent a multiplayer quantum game, for which the 'classical win probability' is one half (as in the parity game), but which can be won with certainty if the players are provided with a 'resource state' which is a 'cat state' superposition of the ground states of Kitaev's two dimensional toric code. I remind the reader that the toric code is defined on a two dimensional square lattice with periodic boundary conditions (i.e. on a torus), and has four degenerate ground states, related to each other by the insertion (or deletion) of a Z_2 'flux' through either of the two holes of the torus. If the different 'topological' sectors are labelled $|ij\rangle$, where $(i, j) = 1(0)$ indicates that a Z_2 flux is present (absent) through the corresponding hole, then the necessary 'resource state' which yields quantum advantage at the task of winning this game, is a cat state of the form $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$. Armed with this resource state, the 'players' can 'win' the game with probability one. This provides an intriguing new perspective on the characterization of topological order. I note however that the resource state needs to be a cat state of toric code ground states at the exactly solvable point (unperturbed toric code). While small perturbations of the Hamiltonian about the solvable point leave the system in the same phase, the ground states of the perturbed Hamiltonian do not appear to yield quantum advantage with respect to the task of winning the game in (1). Furthermore, even when armed with the appropriate resource state, obtaining quantum advantage requires the ability to apply non-local 'square root of Wilson loop' operators which wrap at least one of the cycles of the torus. This may be challenging to carry out in a macroscopic system.

Both these limitations are substantially circumvented in the second of the highlighted papers (2), which introduces a different class of games. These are 'winnable' if armed with a resource state formed from ground states of the transverse field Ising model (TFIM) in one dimension. Specifically, if one denotes by $|000...0\rangle$ and $|111...1\rangle$ the two classical ground states of the TFIM in its symmetry breaking phase (colloquially, 'all spins up' and 'all spins down'), then a resource state which is a cat state superposition of the form $|GHZ^+\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$ allows the game from (2) to be won with better-than-classical probability. Importantly, the quantum advantage is no longer limited to the exactly solvable point. Instead, quantum advantage persists through a large swath of the ordered phase, and indeed the 'boundary of quantum advantage' can be pushed asymptotically close to the phase transition by modifying the game. Additionally, the operations necessary to gain quantum advantage involve only local unitaries and measurements, rather than complicated non-local operations. The price that must be paid, is that one has to work with a cat state of two symmetry broken states, which is not robust to symmetry breaking perturbations.

In combination the highlighted papers present a stimulating new way of thinking about quantum phases of matter, inspired by quantum information, in terms of whether the correlations inherent in the phases can be exploited to gain quantum advantage at a certain task (winning a 'quantum game'). The first paper demonstrates how to accomplish this for the toric code, albeit the protocol only works at the exactly solvable point, and requires the ability to perform non-local operations. The second paper demonstrates how to accomplish this for the transverse field Ising model - this time the protocol works even away from the exactly solvable point, and only requires local operations, but still makes use of 'cat state' inputs which may not be robust to e.g. symmetry breaking perturbations. These two papers should be viewed as the starting point of a new way of thinking about quantum phases of matter and how to characterize and harness the correlations inherent therein.

There are some obvious directions for future investigation. For example, can one come up with ‘games’ which are fully robust? Topological phases might be naively thought to have an advantage here, in that ‘cat state superpositions’ of topologically distinct ground states would be robust to arbitrary local perturbations. Surprisingly, the results of (1,2) suggest the opposite - the ‘topologically ordered’ states yield advantage only at a solvable point, whereas the symmetry broken phases continue to yield advantage even in the presence of (symmetry respecting) perturbations. Is this surprising inversion a fundamental result, or a limitation of the particular ‘game’ discussed in (1)? Can one come up with better games which harness the robustness of topological phases of matter, which continue to work away from the solvable point, and which don’t require such complicated operations to gain quantum advantage? Can one actually implement any such game on a quantum device? Further afield, can one come up with robust ‘games’ characterizing correlations in other phases of matter? A game for characterizing a particular symmetry protected topological (SPT) state was worked out by Daniel and Miyake. Is there a general protocol for the design of optimal games, such that given a particular quantum phase as an input, one can output the optimal game to harness the correlations inherent therein? Can this ‘games’ perspective be used to gain new insights into the correlations in quantum states? Alternatively, can it prove useful for benchmarking quantum devices? I look forward to future exploration of such ideas, and others stimulated by dialog between the condensed matter and quantum information communities.