The curious case of the backbone scaling exponent

Backbone exponent for two-dimensional percolation
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In 1957, when Broadbent and Hammersley [1] published their original manuscript on percolation theory, the motivating and central analogy of that paper was the flow of fluid in a quenched, disordered medium. In other words, what are the paths that conduct from a source to a sink? In the decades since, percolation models have themselves percolated into many scientific fields including mathematics, computer science, statistical physics, condensed-matter physics, astrophysics, cosmology, geophysics, epidemiology, ecology, etc.

The central feature of the model is its continuous phase transition, where arbitrarily long, connected paths first make an appearance. The transition is similar in some respect to the thermal transitions in the Ising, Potts, and other statistical-mechanical models exhibiting 2^{nd}-order phase transitions at their critical points. In all such transitions, we expect emergent scale and conformal symmetry, and we hope to describe critical properties and find critical exponents from the corresponding conformal field theories (CFTs).

However, there is also a substantial difference in that the percolation model addresses directly the geometric and topological properties of clusters of connected bonds or sites. Once the (incipient) infinite connected cluster appears at the transition, one can ask about its (fractal) scaling dimension D and structure in terms of its various interesting subsets and perimeters (boundaries). For example, one can distinguish the *hull* and the *external perimeter* of the cluster, with the corresponding fractal dimensions D_{h} and D_{ep}.

The internal structure of a critical cluster can be probed in the setting where each bond in a cluster represents an electrical conductor, and then one looks at the current flow in a large sample connected to two electrodes, either at two distinct points, at a point and a distal boundary, or at two distant boundary components. In this conductive picture, not all bonds in the cluster actually carry current – a consequence of Kirchhoff's Laws. The ones that do, form the *backbone* of the cluster (of dimension D_{b}). The so-called *red bonds* of the backbone (of dimension D_{r}) carry the maximal current, and if they are cut, the current flow stops. The rest of the cluster is comprised of the *dangling ends* [2–4].

Figure 1, taken from the recent preprint "Backbone exponent for two-dimensional percolation" by Pierre Nolin, Wei Qian, Xin Sun, and Zijie Zhuang [5], shows one 2D critical

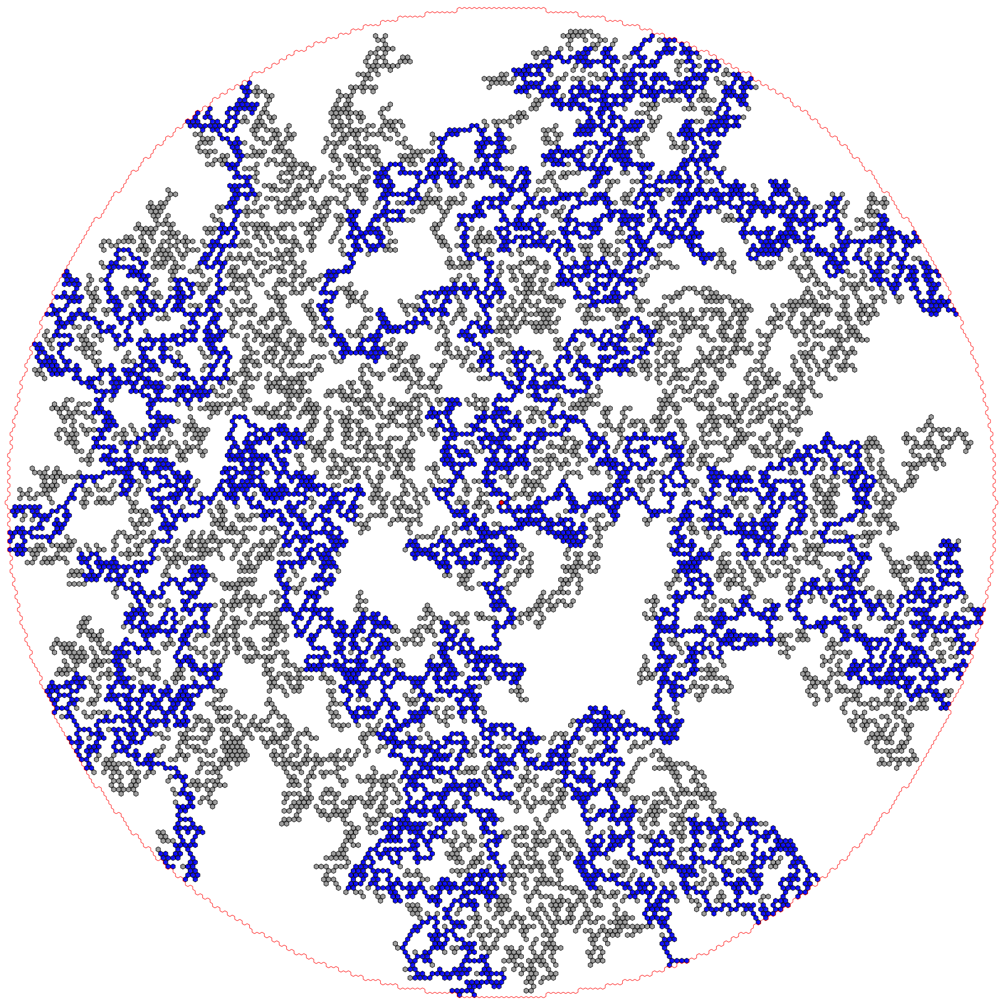

Figure 1: A conductive backbone (blue) in a critical site-percolation cluster

cluster that connects the red dot at the origin (one electrode) to the red, circular boundary (the other electrode). In this figure, the backbone is shown in blue while the rest of the cluster (the dangling ends) is in gray. Looking at the figure, one might surmise that $D_{\mathrm{b}} < D$. In fact, for 2D percolation, it can be shown that $D_{\mathrm{ep}} < D_{\mathrm{b}} < D_{\mathrm{h}} < D$. One property of the critical backbone that can be readily observed in figure 1 is that its mass is dominated by the multiply connected *blobs*, which appear at all sizes, from the lattice scale up to that of the cluster itself [4, 6]. A consequence of the definition of blobs is that they can be "lit up" (along with the red bonds) by moving the electrodes around to different locations in a fixed sample, providing a uniquely "conducive" description of the entire critical cluster using the same objects.

In a CFT description of the percolation transition, all the above fractal dimensions should be related to scaling dimensions of CFT operators, basic numbers that characterize a given CFT. Field-theory methods are especially powerful in two dimensions, and many critical exponents for 2D percolation, including $D = \frac{91}{48}$, $D_{\mathrm{h}} = \frac{7}{4}$, $D_{\mathrm{ep}} = \frac{4}{3}$, and $D_{\mathrm{r}} = \frac{3}{4}$, and many more, are known exactly. Notice that these exponents are all *rational*, reflecting the fact

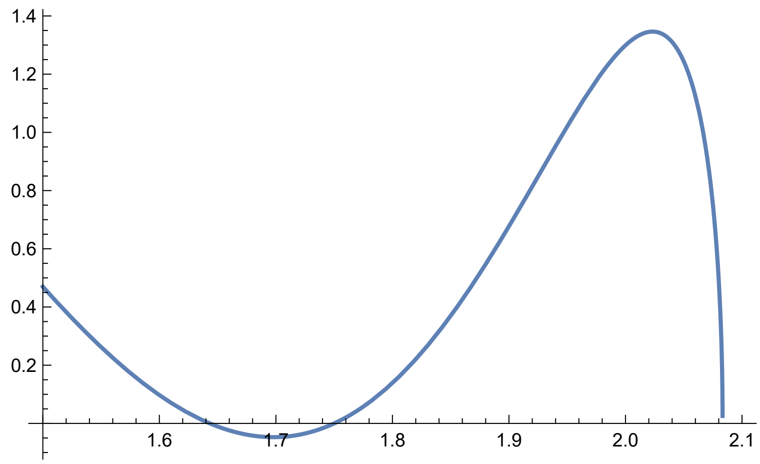

Figure 2: A plot of $y(x)$ versus x on the interval $(1.5, 2.083)$.

that the theory describing 2D critical percolation is an example of a *rational* CFT. Based on this fact, one would expect the backbone dimension D_{b} in 2D to be a rational number as well. Unlike other scaling dimensions, it has *not* been found by CFT methods, and it is also notoriously difficult to obtain numerically [7].* The most recent Monte Carlo estimate [8] is

$$D_{\mathrm{b}} = 1.64339 \pm 0.00005. \tag{1}$$

Against this background, the recent paper [5] appears as a dramatic breakthrough. The striking result of this paper is that D_{b} is the smallest of the three real zeros of the function $y(x)$ (ploted in Fig. 2)

$$y(x) = \sin[\theta\, a(x)] + \frac{1}{2} \sin \theta\, a(x), \tag{2}$$

where $a(x) = \sqrt{25 - 12x}$ and $\theta = 2\pi/3$. The roots of $y(x) = 0$ can be found numerically to any desired accuracy. For the root in question, which lies, as expected, in the interval $\left(D_{\mathrm{ep}} = \frac{4}{3}, D_{\mathrm{h}} = \frac{7}{4}\right)$, we have

$$D_{\mathrm{b}} = 1.643333\,163287\,110417\, \ldots \tag{3}$$

This exact and rigorous result turns out to be not only irrational but *transcendental* as well. Fortunately, and perhaps a sign of things to come, it belongs to a class of compactly expressed transcendentals, about as simple as could be hoped for given the context. Furthermore, Nolin *et al.*'s result will spare future generations of computational physicists the sad and fruitless search for a deep significance of the fraction $493/300 = 1.6433333\ldots$ What of the other two roots of $y(x)$? The next larger root is rational and precisely D_{h}! This is mysterious. Finally, given the form of $a(x)$, the largest root is clearly $x = 25/12$.

*Ziff reports on a conversation with P. Grassberger regarding the total numerical effort $E(n)$ needed to ascertain the n^{th} digit in the decimal expansion of certain percolation exponents, expressed as a factor f multiplying the total effort that went before: $E(n) = f \cdot E(n-1)$. Ziff estimates a lower bound $f \geq 1000$, resulting in (at least) exponential costs with a huge base: $E(n) \geq 1000^n$. R. M. Ziff, personal communication.

We note that this last root is the sum of the external perimeter and red-bond dimensions $D_{\mathrm{ep}} + D_{\mathrm{r}} = 25/12$, but without attaching any meaning to this curiosity.

While we have focused here on the results of Nolin *et al.* applicable to percolation, they can be extended in a straightforward way to all of the q-state Potts models with continuous phase transitions ($q = 0, 1, 2, 3$, and 4); percolation is just the $q = 1$ case. In the general case, the conductive fractions considered are subsets of the Fortuin-Kasteleyn clusters [9]. Of the five backbone exponents $D_{\mathrm{b}}(q)$ associated with those q-state Potts models, the new results indicate that those for $q = 1, 2$, and 3 are transcendental, while only $D_{\mathrm{b}}(0) = 5/4$ and $D_{\mathrm{b}}(4) = 15/8$ are rational. These two values were previously known and are correctly reproduced by the q-analog of Eq. (2).

The recommended paper is the result of a long and spectacular development in what might be called "two-dimensional conformal probability theory". This rigorous approach to 2D critical phenomena and 2D CFT started with a groundbreaking work by Oded Schramm who realized that the boundaries of clusters in critical statistical mechanics models can be characterized precisely by certain stochastic evolutions of conformal maps. This theory, now known as the Schramm-Loewner Evolution (SLE) (see [10–13] for reviews), has led to an explosion of research activity that produced exact, rigorous and conjectural results for many properties of 2D critical systems.

However, in some sense, SLE is "just" an alternative to CFT, and by itself it still failed to predict the exact value of the backbone exponent D_{b}. Another ingredient that proved useful is the connection between critical systems on a plane and those on fluctuating 2D surfaces. This connection, known in physics as 2D quantum gravity (QG), provides the so-called Knizhnik-Polyakov-Zamolodchikov (KPZ) relation between the scaling dimensions of operators in a planar CFT and those in 2D QG [14]. In recent years, this connection has been put on rigorous ground by a number of authors, and has interrelated SLE, conformal loop ensembles, Liouville CFT, large random planar graphs and trees, and more (see [15–17] for reviews). It was this synthesis of different subjects and approaches that allowed Nolin *et al.* to obtain their results (2) and (3). Who could have dreamed of the deep connection between the loopy blobs (or blobby loops) of conducting paths and the quantized fluctuations of a 2D string worldsheet?

There is yet another miracle and mystery lying in the midst of this story. The intermediate steps in the recommended paper are extremely technical, containing numerous complicated hypergeometric and other special functions. Yet, the end result (2) is amazingly simple and elegant. This begs the question (which the authors pose as open): Can the backbone exponent (and other related, so far unknown, exponents) be obtained by more traditional field theoretic and CFT methods? We certainly look forward to more results coming from this beautiful, multiply connected nexus where physics and mathematics meet and interact so fruitfully.

References

[1] S. R. Broadbent and J. M. Hammersley, Proceedings of the Cambridge Philosophical Society **53**, 629 (1957).

[2] A. Coniglio, Phys. Rev. Lett. **46**, 250 (1981), URL https://link.aps.org/doi/10.1103/PhysRevLett.46.250.

[3] A. Coniglio, Journal of Physics A: Mathematical and General **15**, 3829 (1982), URL https://dx.doi.org/10.1088/0305-4470/15/12/032.

[4] H. J. Herrmann and H. E. Stanley, Phys. Rev. Lett. **53**, 1121 (1984), URL https://link.aps.org/doi/10.1103/PhysRevLett.53.1121.

[5] P. Nolin, W. Qian, X. Sun, and Z. Zhuang, arXiv e-prints arXiv:2309.05050 (2023), 2309.05050.

[6] M. F. Gyure, M. V. Ferer, B. F. Edwards, and G. Huber, Phys. Rev. E **51**, 2632 (1995), URL https://link.aps.org/doi/10.1103/PhysRevE.51.2632.

[7] P. Grassberger, Physica A: Statistical Mechanics and its Applications **262**, 251 (1999), ISSN 0378-4371, URL https://www.sciencedirect.com/science/article/pii/S037843719800435X.

[8] S. Fang, D. Ke, W. Zhong, and Y. Deng, Phys. Rev. E **105**, 044122 (2022), 2112.10162.

[9] C. M. Fortuin and P. W. Kasteleyn, Physica **57**, 536 (1972).

[10] W. Kager and B. Nienhuis, Journal of Statistical Physics **115**, 1149 (2004), math-ph/0312056.

[11] J. Cardy, Annals of Physics **318**, 81 (2005).

[12] M. Bauer and D. Bernard, Physics Reports **432**, 115 (2006), math-ph/0602049.

[13] I. A. Gruzberg, Journal of Physics A Mathematical General **39**, 12601 (2006), math-ph/0607046.

[14] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. **A3**, 819 (1988).

[15] R. Rhodes and V. Vargas, arXiv e-prints arXiv:1602.07323 (2016), 1602.07323.

[16] E. Gwynne, arXiv e-prints arXiv:1908.05573 (2019), 1908.05573.

[17] S. Sheffield, arXiv e-prints arXiv:2203.02470 (2022), 2203.02470.