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# Lieb-Schultz-Mattis Theorem for Open Quantum Systems

1. **Lieb-Schultz-Mattis Theorem in Open Quantum Systems**
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   *Phys. Rev. Lett.* **132**, 070402 (2024) and supplemental material

2. **Reviving the Lieb–Schultz–Mattis Theorem in Open Quantum Systems**
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   arXiv:2310.01475

*Recommended with a Commentary by Daniel Arovas, University of California, San Diego*

The Lieb-Schultz-Mattis (LSM) theorem [1] is among the most consequential results in the theory of quantum magnetism. It guarantees that for an antiferromagnetic quantum spin-$S$ chain whose Hamiltonian $H$ is short-ranged, translationally invariant, and has a global $\mathsf{U}(1)$ spin symmetry, the (singlet) ground state is either gapless or doubly degenerate in the thermodynamic limit $L \to \infty$ if $S$ is a half-odd integer. The original (1963) proof utilizes the spin twist operator, $U = \exp(2\pi i \sum_{j=1}^{L} j S_j^z / L)$. One then has $U^\dagger S_n^\pm U = \exp(\mp 2\pi i n/L)\, S_n^\pm$, as well as $U^\dagger t\, U = e^{-2\pi i S}\, t$, where $t$ is the lattice translation operator. If $|\Psi_0\rangle$ is a ground state of $H$, and we define $|\Psi_1\rangle = U\,|\Psi_0\rangle$, then since under the twist $S_n^+ S_{n+1}^- \to e^{2\pi i/L} S_n^+ S_{n+1}^-$, to lowest nontrivial order in $L^{-1}$ and assuming reflection symmetry of $H$,

$$E_1 = \langle\, \Psi_1 \,|\, H \,|\, \Psi_1 \,\rangle = \langle\, \Psi_0 \,|\, H \,|\, \Psi_0 \,\rangle + \frac{2\pi^2}{L^2}\,\langle\, \Psi_0 \,|\, H_\perp^{\rm loc} \,|\, \Psi_0 \,\rangle = E_0 + \mathcal{O}(L^{-1}) \quad , \tag{1}$$

where $H_\perp^{\rm loc}$ is local. Furthermore, if $t\,|\Psi_0\rangle = e^{iK_0}\,|\Psi_0\rangle$, then $t\,|\Psi_1\rangle = e^{iK_1}\,|\Psi_1\rangle$ with $K_1 = K_0 - 2\pi S$. Thus, if $S \in \mathbb{Z} + \frac{1}{2}$, we have $\langle\, \Psi_1 \,|\, \Psi_0 \,\rangle = 0$ because $|\Psi_0\rangle$ and $|\Psi_1\rangle$ have inequivalent crystal momenta. Ta da!

A straightforward generalization to any higher dimension $d > 1$ fails because the bound one obtains is $E_1 \le E_0 + \mathcal{O}(L^{d-2})$. However, Oshikawa's flux-threading argument charts a way forward. Place the system on a $d$-torus, and adiabatically thread a $\mathsf{U}(1)$ flux $\phi$ through one of its cycles. This may be done in a translationally invariant way[1] so that

[1]One replaces $S_{\boldsymbol{r}}^+ S_{\boldsymbol{r}'}^- \to S_{\boldsymbol{r}}^+ S_{\boldsymbol{r}'}^- \exp\left\{ i \int_{\boldsymbol{r}}^{\boldsymbol{r}'} d\boldsymbol{\ell} \cdot \boldsymbol{A}_\phi(\boldsymbol{\ell}) \right\}$, where $\boldsymbol{A}_\phi(\boldsymbol{\ell}) = \phi\,\hat{\mathbf{e}}/L$ is the vector potential of a geometric flux tube threading a cycle of the $d$-torus in the $\hat{\mathbf{e}}$ direction.

$\big[H(\phi), t\big] = 0$ and crystal momentum is preserved throughout the adiabatic flux insertion process. If $|\Psi_0(\phi)\rangle$ is an adiabatic ground state of $H(\phi)$, one can 'pull back' from the Hilbert space of $H(2\pi)$ to that of $H(0)$ via the large gauge transformation $U^\dagger$ using the LSM spin twist operator, defining $|\Psi_1\rangle = U^\dagger\,|\Psi_0(2\pi)\rangle$. One then finds $\boldsymbol{K}_1 - \boldsymbol{K}_0 = 2\pi N_\perp S\,\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is the unit vector in the direction of the flux insertion, and $N_\perp$ is the number of lattice sites in the hyperplane perpendicular to $\hat{\mathbf{e}}$. If $N_\perp$ is odd, then LSM follows for $S \in \mathbb{Z} + \frac{1}{2}$. One assumes (unproven) that this holds for any system dimensions in the thermodynamic limit[2].

In recent years there has been a great deal of activity in studying open quantum systems, such as in cases where a quantum system is in contact with a bath with which it may exchange energy, particle number, magnetization, *etc.* In this context, the system is described not by a wavefunction, but rather by a density matrix. Is there a generalization of LSM to open quantum systems?

Kawabata, Sohal, and Ryu (KSR) [2] considered quantum systems governed by the GKLS master equation [4],

$$\frac{d\varrho}{dt} = \mathcal{L}\varrho = -i[H,\varrho] + \sum_n \Big(L_n\,\varrho\,L_n^\dagger - \tfrac{1}{2}\,L_n^\dagger L_n\,\varrho - \tfrac{1}{2}\varrho\,L_n^\dagger L_n\Big) \quad , \tag{2}$$

where $\varrho(t)$ is the reduced density matrix of the system, $H$ is the Hamiltonian, and the $\{L_n\}$ are 'jump operators' which embody the effects of the environment. The quantum Liouvillean superoperator $\mathcal{L}$ generates a map $\mathcal{C}_t = \exp(\mathcal{L}t)$ which acts on density matrices, such that $C_t\,\varrho(0) = \varrho(t)$. The map $\mathcal{C}_t$ is (i) linear, (ii) trace-preserving, (iii) hermiticity-preserving, and (iv) completely positive[3]. Trace preservation entails that there is at least one (and possibly more, under nongeneric circumstances [5]) nonequilibrium steady state (NESS) $\varrho_0$ satisfying $\mathcal{L}\varrho_0 = 0$. In any basis, the density matrix $\varrho = \sum_{\alpha,\beta} \varrho_{\alpha\beta}\,|\alpha\rangle\langle\beta|$ may be expressed as a vector $|\varrho\rangle = \sum_{\alpha,\beta} \varrho_{\alpha\beta}\,|\alpha\rangle \otimes |\beta\rangle$, a manipulation known as the Choi-Jamiołkowski isomorphism. Eqn. 2 may then be recast as a non-Hermitian Schrödinger equation, $i\,d\,|\varrho\rangle/dt = \mathcal{H}|\,\varrho\,\rangle$, with $\mathcal{H}$ acting on a doubled Hilbert space, *viz.*

$$\mathcal{H} = H \otimes I - I \otimes H + i\sum_n \Big(L_n \otimes L_n^* - \tfrac{1}{2}\,L_n^\dagger L_n \otimes I - I \otimes \tfrac{1}{2}\,L_n^\intercal L_n^*\Big) \quad , \tag{3}$$

where $I$ is the identity[4]. The eigenvalues $E_a$ of $\mathcal{H}$ all satisfy $\gamma_a = -\mathsf{Im}(E_a) \geq 0$, where $\{\gamma_a\}$ is the spectrum of relaxation rates. Any NESS has $E_a = 0$.

One may now define two types of symmetries [7]. A *strong symmetry* is one which commutes with $H$ and all the $L_n$, whereas a *weak symmetry* commutes only with $\mathcal{L}$ as a whole. KSR consider a system with translational invariance and strong $\mathsf{U}(1)$ symmetries

[2]More generally, Oshikawa showed that if the $\mathsf{U}(1)$ charge per unit cell is $\nu = p/q$ with $p$ and $q$ relatively prime, then a unique gapped ground state cannot exist when $\nu N_\perp \neq \mathbb{Z}$. To elicit a gap then requires a breaking of translational symmetry in which the unit cell is $q$-fold enlarged. For quantum spins, the local $\mathsf{U}(1)$ charge is $n_j = S_j^z + S$.

[3]A map $\mathcal{C}$ is completely positive if $(\mathcal{C}\varrho) \otimes \omega$ is positive whenever $\varrho \otimes \omega$ is positive. Positivity means that $\langle\Psi|\,\varrho\,|\Psi\rangle \geq 0$ for all $|\Psi\rangle$. An example of a linear map which preserves trace and hermiticity yet which is not completely positive is matrix transposition.

[4]There is no new physics revealed by expressing Eqn. 2 in this way, but it perhaps provides a more familiar setting, as well as entailing some notational conveniences.

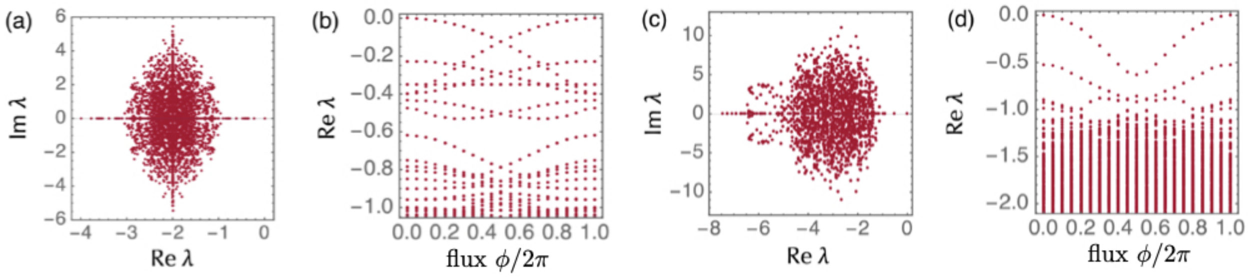


Figure 1: Fig. 1 from Ref. [2]. Eigenvalues $\lambda$ of the dissipative XXZ quantum Liouvillean $\mathcal{L}(\phi)$ in the Heisenberg limit $\varDelta = \eta = 1$ and total filling $\nu = \frac{1}{2}$, *i.e.* $S^z_{\text{tot}} = 0$. ($E = i\lambda$ are the eigenvalues of $\mathcal{H}$ in Eqn. 3; the relaxation rates are $\gamma = -\mathsf{Re}\,\lambda \geq 0$.) (a,b) $S = \frac{1}{2}$, $L = 8$. (c,d) $S = 1$, $L = 5$. In both cases the $\mathsf{U}(1)$ flux is adiabatically inserted in the first (+) Hilbert space. For all flux $\phi$ there is an infinite temperature NESS with $\lambda = 0$ (not shown in panels b and d).

$\mathcal{U}_\pm$, where $\mathcal{U}_+ = U_+ \otimes I$ and $\mathcal{U}_- = I \otimes U_-$. (Translational invariance is a weak symmetry.) The unitary symmetries guarantee conserved charges $N_\pm$ in the two Hilbert spaces, where $N_\pm = \sum_j (S^z_{j,\pm} + S)$. Assuming $N_+ = N_-$, they show that if $\nu \equiv N_\pm/V$ is not an integer ($V$ is the total number of spins), then $\mathcal{L}$ is either gapless or exhibits degenerate NESS. The proof directly follows Oshikawa's derivation of LSM. Results for the $XXZ$ Hamiltonian

$$H = \sum_{n=1}^{L} \Big( S^x_n\, S^x_{n+1} + S^y_n\, S^y_{n+1} + \varDelta\, S^z_n\, S^z_{n+1} \Big) \tag{4}$$

with jump operators $L_n = \sqrt{\eta}\, S^z_n$ are shown in Fig. 1. A $\mathsf{U}(1)$ flux $\phi_\pm$ may be inserted in either Hilbert space in a translationally-invariant way with periodic boundary conditions (see footnote 1 above), or by imposing twisted boundary conditions $S^+_{n+L,\pm} = \exp(i\phi_\pm)\, S^+_{n,\pm}$, which breaks translational invariance. Because all jump operators are Hermitian, it is easy to see that the infinite temperature state $\varrho_0 = 2^{-L} I$ is a NESS for any $\phi$. Panels (a,b) show results for a dissipative $S = \frac{1}{2}$ chain, while panels (c,d) correspond to $S = 1$. Each dot corresponds to an eigenvalue of $\mathcal{L}(\phi) = -i\mathcal{H}(\phi)$. For $S = \frac{1}{2}$, the spectral flow in panel (b) shows crossings at the time-reversal symmetric value $\phi = \pi$, whereas for $S = 1$ panel (d) there are no such crossings. The numerics are for small system sizes ($L = 8$ and $L = 5$), and presumably as $L \to \infty$ this flow results in a dense set of states in the vicinity of $\lambda = 0$ for $S = \frac{1}{2}$, and that this is not the case for $S = 1$. Such a scenario would be consistent with KSR's version of LSM. Much more is explored in the body of KSR as well as in the supplemental material, but on to paper #2.

In Ref. [3], Zhou *et al.* approach the issue from the perspective of the entanglement Hamiltonian $K \equiv -\ln \varrho_{\mathsf{s}}$ [8], where $\varrho_{\mathsf{s}} = \mathsf{Tr}_{\mathsf{b}}\big(|\Psi_0\rangle\langle\Psi_0|\big)$ is the reduced density matrix of a system *explicitly* coupled to an environmental bath, rather than obtained as a NESS from GKLS dynamics, which requires some phenomenological choice of the jump operators[5]; $|\Psi_0\rangle$ is the ground state of the system plus bath. Two conditions are imposed. First, both discrete

[5]Deriving the jump operators from the system, bath, and system-bath coupling Hamiltonians is formally described in ch. 3 of Breuer and Petruccione [4], but is in general a highly impractical procedure.

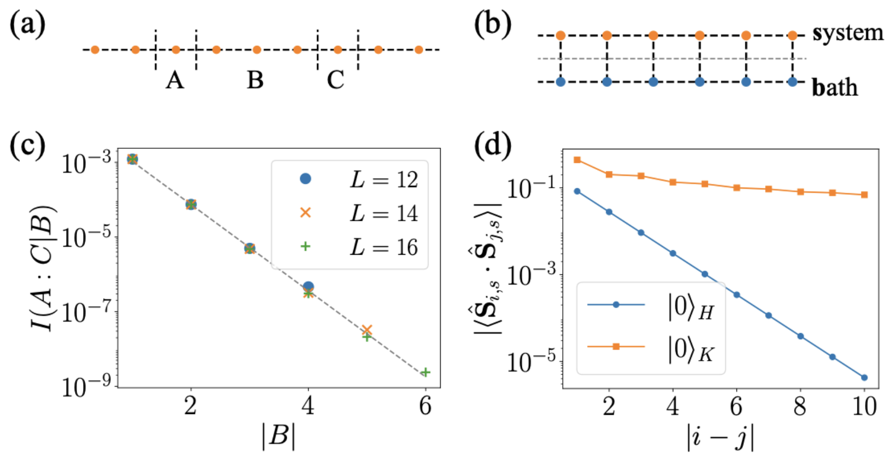


Figure 2: Fig. 2 from Ref. [3]. (a) Tripartition of a spin chain with regions $\mathsf{A}$, $\mathsf{B}$, and $\mathsf{C}$. (b) System-bath scenario. The upper chain (system) spins have $S_{\mathsf{s}} = \frac{1}{2}$ while the lower chain (bath) spins have $S_{\mathsf{b}} = \frac{3}{2}$. (c) Mutual information $I(\mathsf{A} : \mathsf{C}\,|\,\mathsf{B})$ as a function of the size $|\mathsf{B}|$. (d) Magnitude of the system spin-spin correlation $|\langle \boldsymbol{S}_{i,\mathsf{s}} \cdot \boldsymbol{S}_{j,\mathsf{s}} \rangle|$ *versus* separation $|i-j|$ in the ground states $|0\rangle_H$ of the physical Hamiltonian and $|0\rangle_K$ of the entanglement Hamiltonian.

translation and continuous spin rotation are weak symmetries, with $U^\dagger K U = K$. That is to say, the total state of the system plus bath is invariant under these symmetries. Second, it is assumed that the system spins are short-range correlated due to the coupling to the bath, with exponential decay of system spin correlations $C_{ij} = \mathsf{Tr}\big(\varrho\, S^z_{i,\mathsf{s}} S^z_{j,\mathsf{s}}\big)$ on a length scale $\xi$ which could arise if the bath opens up a gap $\varDelta \sim \xi^{-1}$ or if the system effectively thermalizes at some temperature $T$.

The intuition is that the system is in an 'approximate quantum Markov state' where the conditional mutual information shared by regions $\mathsf{A}$ and $\mathsf{C}$ separated by $\mathsf{B}$ (see Fig. 2(a)) is small, *i.e.* $I(\mathsf{A} : \mathsf{C}\,|\,\mathsf{B}) < \varepsilon$ with $\varepsilon$ vanishing as $\exp(l_{\mathsf{B}}/\xi)$ where $l_{\mathsf{B}}$, is the width of region $\mathsf{B}$. The conditional mutual information is defined as $I(\mathsf{A} : \mathsf{C}\,|\,\mathsf{B}) = I(\mathsf{A} : \mathsf{BC}) - I(\mathsf{A} : \mathsf{B})$, where $I(\mathsf{A} : \mathsf{B}) = S_{\mathsf{A}} + S_{\mathsf{B}} - S_{\mathsf{AB}}$ is the mutual information shared by $\mathsf{A}$ and $\mathsf{B}$, and $S_{\mathsf{A}}$ is the von Neumann entropy of $\mathsf{A}$[6]. When $I(\mathsf{A} : \mathsf{C}\,|\,\mathsf{B})$ is small, it means that $\mathsf{B}$ inhibits information-sharing between $\mathsf{A}$ and $\mathsf{C}$, and when $I(\mathsf{A} : \mathsf{C}\,|\,\mathsf{B}) = 0$, then $K_{\mathsf{ABC}} = K_{\mathsf{AB}} + K_{\mathsf{BC}} - K_{\mathsf{B}}$, which says that the entanglement Hamiltonian for the entire region $\mathsf{ABC}$ may be broken up into contributions from smaller regions. Zhou *et al.* conjecture that when the system is in an approximate quantum Markov state that $K$ is exponentially local. This permits an estimate of the 'energy' difference $\Delta = \langle U^\dagger K U\rangle_0 - \langle K\rangle_0$, where $U$ is the LSM twist operator and the expectation values are taken in the ground state $|0\rangle_K$ of $K$. A simple calculation yields this difference $\Delta \sim 4\pi^2\xi^3/L$ as $L \to \infty$. Since $K$ is translationally invariant, $|0\rangle_K$ and $U|0\rangle_K$ have inequivalent crystal momenta if the system spins are all $S \in \mathbb{Z} + \frac{1}{2}$, by the LSM argument.

Zhou *et al.* study two explicit models. The first is given by

$$H = \sum_{j=1}^{L} \Big( \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \tfrac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 + \gamma\, \boldsymbol{S}_{j,\mathsf{s}} \cdot \boldsymbol{S}_{j,\mathsf{b}} \Big) \quad , \tag{5}$$

[6] For example, if $\mathsf{A}$ and $\mathsf{B}$ are disconnected, then $S_{\mathsf{AB}} = S_{\mathsf{A}} + S_{\mathsf{B}}$ and $I(\mathsf{A} : \mathsf{B}) = 0$.

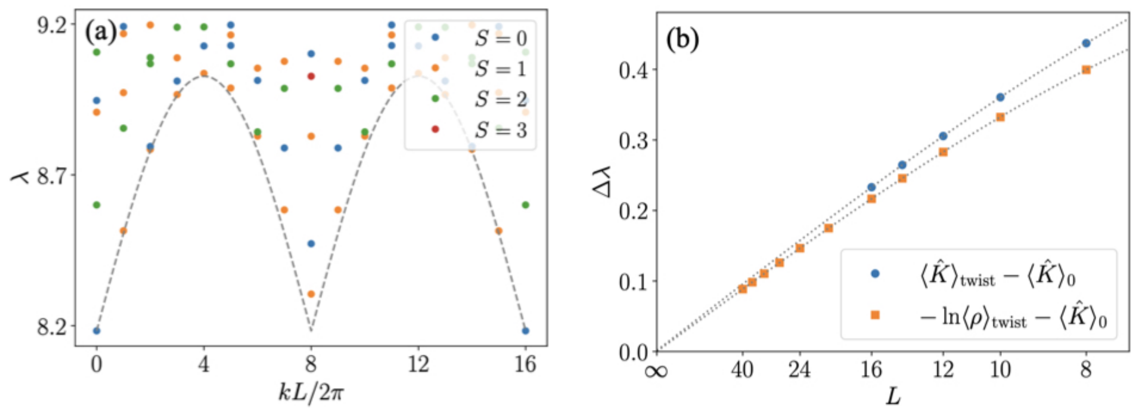


Figure 3: Fig. 3 from Ref. [3]. (a) Total spin and momentum resolved entanglement spectrum. The dashed line is given by $\lambda = v|\sin k| + \lambda_0$, with $v$ and $\lambda_0$ fitted from the lowest eigenvalues at the lowest two momenta and $L = 16$. (b) Comparison of $\langle U^\dagger K U\rangle_0 - \langle K\rangle_0$ (blue) and $-\ln\langle U^\dagger \varrho\, U\rangle_0 - \langle K\rangle_0$ (orange) *versus* $L$. See text for more details.

where $S_{\mathsf{s}} = \frac{1}{2}$ and $S_{\mathsf{b}} = \frac{3}{2}$, and $\boldsymbol{S}_j \equiv \boldsymbol{S}_{j,\mathsf{s}} + \boldsymbol{S}_{j,\mathsf{b}}$. This corresponds to the spin-ladder depicted in Fig. 2(b). When $\gamma > 0$, each rung of the ladder is preferentially in a total spin $S = 1$ state, and in this subspace remaining terms in $H$ yield the $S = 1$ AKLT Hamiltonian [9], which exhibits a Haldane gap [10] and a spin correlation length of $\xi_{\rm AKLT} = 1/\ln 3 \approx 0.91$. Given the exact many-body ground state in the large-$\gamma$ limit, one can numerically compute the entanglement Hamiltonian $K$ using singular value decomposition. Fig. 2(c) shows the mutual information $I(\mathsf{A}:\mathsf{C}\,|\,\mathsf{B})$, which exhibits a clear exponential decay consistent with the approximate quantum Markov state assumption. The decay length for the chosen value of $\gamma$ ($= 1$?) is $\xi \approx 0.38$, which is comparable though smaller (as expected) to the AKLT result. In Fig. 3(a), the spin and momentum resolved eigenvalues $\lambda$ of the entanglement Hamiltonian $K$ are shown ($L = 16$). In Fig. 3(b) the difference $\langle U^\dagger K U\rangle_0 - \langle K\rangle_0$ is plotted *versus* $L$. It is well-fit to the curve $c_0 + c_1/L + c_3/L^3$ with $(c_0, c_1, c_3) = (0.0014, 3.8, -19)$[7]. The tiny values of $c_0$ suggest that the entanglement spectral gap vanishes in the thermodynamic limit. In addition to the Hamiltonian of Eqn. 4, another model is studied: a $S = \frac{1}{2}$ Majumdar-Ghosh chain [11] coupled to a $S = \frac{3}{2}$ bath, which yields a doubly degenerate entanglement Hamiltonian ground state separated by a robust gap from the other eigenvalues, again consistent with LSM.

It would have been illustrative to consider a $S_{\mathsf{s}} = 1$ chain coupled to a bath and contrast the results with those of the two $S_{\mathsf{s}} = \frac{1}{2}$ chains studied. It would seem that a form of Oshikawa's flux insertion argument should be applicable, extending the conclusions of Zhou *et al.* to dimensions $d > 1$.

I am grateful to Yi-Zhuang You for helpful discussions and for alerting me to Ref. [3].

[7] A second difference, $-\ln\langle U^\dagger \varrho\, U\rangle_0 - \langle K\rangle_0$, which is numerically easier to compute, is also plotted.

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