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Symmetries without an inverse: An illustration through the 1+1-D Ising model

1. **Majorana chain and Ising model -- (non-invertible) translations, anomalies, and emanant symmetries**
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   https://arxiv.org/abs/2307.02534

Recommended with a Commentary by T. Senthil, Massachusetts Institute of Technology

Symmetries provide a powerful tool to organize our understanding of physics. In recent years, the notion of symmetry in quantum many body systems/field theory has been generalized in many interesting ways, resulting in a unified perspective through which many phenomena can be discussed. Of particular interest for this commentary is a generalization to symmetry operations that have no inverse.

Ordinary symmetry operations form a group. Thus every symmetry operation has an inverse. In a quantum system, these symmetries are described by unitary (or sometimes, as in the case of time reversal, anti-unitary) operators that commute with the Hamiltonian. In recent years however the concept of non-invertible symmetries has been explored. These symmetries can not be described by unitary operators in quantum systems. The operators that correspond to these symmetries commute with the Hamiltonian. In the last 2 years, it has become clear that many quantum many body systems/field theories in spatial dimension d > 1 have such non-invertible symmetries. Just like ordinary symmetries, they provide a powerful tool for organizing our understanding of the physics.

The general idea is illustrated very concretely in the recommended paper with an example that is completely familiar to condensed matter physicists: the transverse field Ising model in one space dimension[1]. This model describes a system of spin-1/2 moments (qubits) in a chain of N sites with the Hamiltonian

$$H = -g\sum_{j=1}^{N} Z_j - \sum_{j=1}^{N} X_j X_{j+1}$$

The $\left(X_j, Z_j\right)$ are the usual Pauli operators that act on the spin/qubit at site j. (The Hilbert space of the full system is the tensor product of the Hilbert spaces of each qubit and is 2^N dimensional). Consider periodic boundary conditions so that $X_{N+1} = X_1, Z_{N+1} = Z_N$.

The model obviously has a global Ising spin flip symmetry corresponding to π rotation of each spin about the Z-axis, in addition to lattice translation symmetry. The Ising symmetry is implemented by the operator

$$\eta = \prod_{j=1}^{N} Z_j$$

When g > 1, the ground state is paramagnetic and the Ising symmetry is unbroken. For g < 1, in the thermodynamic limit ($N \to \infty$), there are two degenerate ground states that correspond to the two symmetry broken ground states with ferromagnetic ordering along the X-direction. At g = 1, there is a quantum critical point (which is described by the same fixed point theory as the classical 2d Ising model).

It is also well-known that the model has a self-duality property, first described by Kramers and Wannier[2] for the related 2d Ising model in 1941. In the context of the transverse field Ising chain, it (roughly) corresponds to the transformation

$$Z_j \leftrightarrow X_j X_{j+1},$$

under which the critical point at g = 1 remains invariant. But, in what sense, is the Kramers-Wannier duality a symmetry of the model? Suppose we try to invent a unitary operator U that implements the duality transformation. Then U needs to satisfy

$$U Z_j U^\dagger = X_j X_{j+1}$$

However this implies that $U \eta U^\dagger = U \left(\prod_{j=1}^{N} Z_j \right) U^\dagger = \prod_{j=1}^{N} (X_j X_{j+1}) = 1$, which is equivalent to saying that $\eta = 1$ which is manifestly incorrect. Thus we conclude that there is no unitary symmetry operator in the Hilbert space that implements the duality transformation.

However there is a non-invertible operator that acts on the physical Hilbert space, commutes with the Hamiltonian, and implements the duality transformation. This operator is found explicitly in the selected paper, and is given by

$$D = e^{-\frac{2\pi i N}{8}} \left(\prod_{j=1}^{N-1} \frac{1 + iZ_j}{\sqrt{2}} \frac{1 + iX_j X_{j+1}}{\sqrt{2}} \right) \left(\frac{1 + iZ_N}{2} \right) \frac{1 + \eta}{2}$$

The reader is invited to check that this operator is (a) translation invariant, (b) commutes with the critical g = 1 Hamiltonian $HD = DH$, and (c) implements the duality transformation through the action

$$D X_j X_{j+1} = Z_{j+1} D, \quad D Z_j = X_j X_{j+1} D, \text{ for j = 1,…, N.}$$

However note that while the first two factors in parentheses in the definition of D are unitary, the last factor $\frac{1+\eta}{2}$ is a projector that annihilates all states odd under the global Ising symmetry. This makes D a non-invertible operator. The recommended paper contains a derivation of the form of D through the relationship of the transverse field Ising model to the Majorana chain (via the Jordan-Wigner transformation). While many individual aspects of the discussion will be familiar to condensed matter physicists, they are put together in a way that leads to interesting conclusions.

To summarize, it is possible to find a conserved operator that implements the duality transformation of the critical Ising chain but it is not invertible. Thus the self-duality of the Ising model is properly viewed as a non-invertible symmetry. As is well-known the self duality is a very useful tool to understand the physics of the critical Ising model which we can now restate as a consequence of this non-invertible symmetry. Further discussion beyond the recommended paper is in Ref. [5].

This example in the Ising model can serve as a portal for condensed matter physicists into the much wider emerging literature on non-invertible symmetries in quantum many body physics and quantum field theory. Such symmetries have been found in a wide variety of models, including some familiar ones[3,4]. For instance, consider the chiral anomaly of the massless Dirac fermion in 3 space dimensions (possibly known to condensed matter physicists through its manifestation in Dirac materials) where the axial current is not conserved in the presence of a probe electromagnetic field. This can be reinterpreted as a true symmetry involving the axial current (even in the presence of the electromagnetic field) but one that is non-invertible[5,6].

As a potentially new non-perturbative tool, it seems likely that the study of such unusual generalizations of symmetry will play an important role in shaping our understanding of quantum many body physics/field theory in the future.

Apart from the recommended paper, the reader might find the reviews in Refs. [3,4] useful.

References:

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