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Leggett's bounds or how to quantify superfluidity

1. **Superfluid fraction in an interacting spatially modulated Bose-Einstein condensate**
   Authors: R G. Chauveau, C. Maury, F. Rabec, C. Heintze, G. Brochier, S. Nascimbene, J. Dalibard, J. Beugnon, S. M. Roccuzzo, S. Stringari
   arXiv:2302.01776
   and Phys. Rev. Lett. **130** 226003 (2023)
2. **Observation of anisotropic superfluid density in an artificial crystal**
   Authors: Junheng Tao, Mingshu Zhao, Ian Spielman
   arXiv:2301.01258
   and Phys. Rev. Lett. **131** 163401 (2023)
3. **Superfluid fraction of interacting bosonic gases**
   Authors: Daniel Pérez-Cruz, Grigori E. Astrakharchik, Pietro Massignan
   arXiv:2403.08416

Recommended with a Commentary by Thierry Giamarchi, University of Geneva

How to characterize superfluidity (or indifferently superconductivity) is an important question. The most naive answer is the existence of an order parameter below which the $U(1)$ phase symmetry of the wavefunction ψ is spontaneously broken such that $\langle\psi(x)\rangle$ acquires a finite value. However we know that the situation is actually more subtle, and that superfluidity can also exist in the absence of an order parameter. In addition, in anisotropic systems, or in presence of an external potential, the superfluid fraction (superfluid density normalized to the total density) is actually not a scalar but depends on the direction. A simple way to define it is the linear response to an applied force [1]

$$f_{\alpha,\beta} = 1 - \lim_{v_\beta \to 0} \frac{\langle P_\alpha \rangle}{N m v_\beta} \tag{1}$$

where P_α is the momentum in the direction α, N the number of particles, m the mass and the system is subjected to a perturbation moving at velocity v_β in the direction β. Looking at properties at zero temperature this is related to the famous Kohn stiffness [2] relating the second derivative of the ground state energy with respect to a gauge potential in the direction α, which measures also, quite logically for superfluidity, the weight of the Drude peak (zero frequency conductivity). There are some subtleties when the temperature is finite between energy and free energy [3] that I will mostly ignore since this commentary mostly focuss on zero temperature. In practice this last expression allows for direct calculations of the superfluid fraction, e.g. with methods such as quantum Monte-Carlo by measuring the winding number, and thus probe the effects of the interactions or potential on this essential quantity. However these exact methods of computing the superfluid stiffness are quite involved and require either powerful numerical techniques of analytical ones to evaluate. Moreover they might need input that is not necessarily easy to measure either in a condensed matter or in a cold atomic setup.

To have another access the superfluid stiffness, in a set of two remarkable papers Leggett has devised much simpler, although not rigorous, estimates of the superfluid density resting on the knowledge of the *density* alone. A first paper [4] defines an upper bound, detailed below for the case of a plane (for simplicity) with two orthogonal coordinates x and y.

$$f_{\mathrm{Max}} = \frac{1}{\rho_0 \left\langle \frac{1}{\langle \rho(x,y) \rangle_y} \right\rangle_x} \tag{2}$$

where ρ_0 is the average density and $\langle \cdots \rangle_\nu$ denotes an averaging of the density along the corresponding direction. So for example

$$\langle \rho(x,y) \rangle_y = \frac{1}{L_y} \int dy \langle \rho(x,y) \rangle \tag{3}$$

where L_y is the size in the direction y and $\langle \cdots \rangle$ is the usual quantum average. The idea behind such an upper bound (see the Fig. 1 of paper 3) is that places in the system in which the density is very small are blocking points for passing a current and thus should limit the superfluid fraction. The quantity is simple and easy to measure if one has access to the density at each point, which is the case with state of the art cold atom experiments.

A second, and much later paper [5] introduced a different but related quantity which was deemed to be providing a lower bound. It is given by

$$f_{\mathrm{heur}} = \left\langle \frac{1}{\rho_0 \langle \frac{1}{\rho(x,y)} \rangle_x} \right\rangle_y \tag{4}$$

and corresponds to a different way to weight the depleted regions. If (4) was a faithful lower bound one would thus have a very useful tool to deal with the question of superfluidity. I'll come back to this question below, but one can see immediately that if the density factorizes $\rho(x,y) = n(x)n(y)$ then the two quantities are straightforwardly equal, showing that the quantity f_{heur} is unlikely to be a faithful lower bound. In any case as usual with bounds the most important question after the one of their existence, is whether they are or not efficient bounds.

In that respect one can find counter examples when the interactions are strong. This can easily be seen in a one dimensional situation in which it is well known that an infinitesimally small periodic potential of the right period (one boson per period) can lead to a gapped Mott insulator state for the bosons. This prediction of the Tomonaga-Luttinger liquid physics has been beautifully confirmed by experiments [6, 7]. In such cases the density is essentially homogenous leading to (4) being around one, while the superfluid density is obviously zero. Note that in that case the upper bound, although correct as a bound, is not a very stringent limitation on the superfluid density and certainly far from the actual value. A similar case (also noted on paper 3) would occur for disordered systems which in 1D can lead to a localized bose Glass (thus with zero superfluid fraction) with an infinitesimal disorder (thus with a practically uniform density) when the interaction is repulsive enough.

So in order to test the quality of these two estimators based on density, two recent experimental papers and one theoretical one, using in particular numerical simulations on both clean and disordered bosonic systems compared the exact values for the superfluid fraction with these "bounds". Both the experimental papers put a periodic potential modulated in a single direction $V(x, y) = V_0 \cos(Qx)$ and measured the superfluid fraction (superfluid density normalized to the total density), both along the direction x and the direction y. Such a potential leads naturally to the above mentioned factorizable density and thus to (2) and (4) to be equal. The measurement itself is interesting and is done by measuring essentially the sound velocities along x and y which are related quite generally to the superfluid fraction (see Fig. 2c of paper 2 and Fig. 1 of paper 1). These quantities are also computed using a Gross-Pitaevskii [1] calculation appropriate for weakly interacting bosonic gases.

The results are remarkable in themselves and show clearly the effect of the periodic potential on the depletion of the superfluid fraction. When the superfluid fraction is compared with the Leggett "bounds" (see Fig. 2d of paper 2 and Fig. 1 of paper 1) the comparison is spectacularly good with in both case a quasi-perfect agreement between the experimental reduction of superfluid density and the upper bound. In paper 3 more complex cases with a modulation both along x and along y of potential are examined numerically. In that case the agreement is again excellent (see Fig. 2 of paper 3). The case of disordered potentials examined numerically in paper 3 shows a more complex situation. In that case f_{Max} and f_{heur} do not coincide. Nevertheless the two quantities are in the relative order expected for a upper and a lower bound and the numerically computed exact superfluid fraction has the same general shape and is well within the two "bounds" (see Fig 4 of paper 3).

Thus for weakly coupled gases, these simple estimators based on density show a surprising level of success in describing the superfluidity. This directly prompts for further analysis of why and when we can expect such indicators to be good, or how we could perfect them in situations in which they fail, without going to a full calculation or measurement of the superfluid density. One can certainly speculate that one would thus be largely protected when the blocking of the current is mostly connected to "classical" effects but could fail if more subtle correlations and interferences are responsible for the destruction of superfluidity. Disordered systems should thus be particularly informative in that respect. Whatever will be the final word on this subject, understanding better such estimators is clearly an interesting and challenging problem that this will stimulate further analysis and experiments in the future.

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