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A new "framing" of non-collinear antiferromagnetism

Spontaneous symmetry breaking in the Heisenberg antiferromagnet on a triangular lattice
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Field theory is a common language for describing many-body systems that arise in particle physics, condensed matter, and statistical mechanics, and often allows ideas to be transferred fruitfully between these subdisciplines. This is particularly salient in the study of classical and quantum magnetism, where low-energy, long-wavelength theories for coarse-grained order parameter fields in space-time often take the form of "nonlinear sigma models" (NLSMs), that are familiar from particle physics and string theory. A textbook [1] example is furnished by the bipartite Heisenberg antiferromagnet, where the order parameter — the Néel vector — describes the difference in magnetization between the two sublattices, $\boldsymbol{n} = \frac{1}{2}(\boldsymbol{m}_1 - \boldsymbol{m}_2)$. The corresponding continuum field theory has the form of an NLSM for a vector $\boldsymbol{n} \in \mathcal{S}^2$:

$$\mathcal{L} = \frac{\rho}{2}\partial_t \boldsymbol{n} \cdot \partial_t \boldsymbol{n} - \frac{\mathcal{J}}{2}\partial_i \boldsymbol{n} \cdot \partial_i \boldsymbol{n}, \tag{1}$$

where the nonlinearity stems from the constraint $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ required by the $O(3)$ symmetry. Eq. (1) can be obtained formally via a semiclassical path-integral treatment of quantum spins S with local Heisenberg exchange, where the small parameter is $1/S$ [2].

A key use of the NLSM is to understand the spontaneous breaking of the global symmetry, and to organize the resulting spectrum of gapless Goldstone modes. For the Néel antiferromagnet, where the spins are collinear, the story is well-known: there are two Goldstone modes, or magnons, with identical speeds, and both the counting and the degenerate dispersion can be viewed as a consequence of the unbroken symmetry of rotation about the collinear axis. Richer behaviour can occur in systems where lattice geometry frustrates Néel order in favor of more complicated configurations. A case in point is the triangular lattice Heisenberg model, where the (semiclassical) ground state configuration is non-collinear and involves three sublattices. In this case, there is no obvious residual symmetry, suggesting there should be three Goldstone modes all with independent speeds. As it turns out, while the counting is correct, two of the three modes are degenerate. Rationalizing this subtle feature is the central focus of the recommended paper, and involves teasing out a "hidden symmetry" via clever methods that might find purchase on a wider array of problems.

Before proceeding to the more challenging three-sublattice problem, let us first examine the relatively familiar two-sublattice antiferromagnet in some more detail. The ground state Néel vector $\boldsymbol{n}_0$ is uniform in real space and points in an arbitrary direction in three-dimensional spin space, spontaneously breaking the global $SO(3)$ symmetry of Eq. (1) down to $SO(2)$, corresponding to rotations around $\boldsymbol{n}_0$. A standard calculation reveals a pair of magnons corresponding to small transverse deviations of the spin order parameter from $\boldsymbol{n}_0$. Since the ground state breaks the symmetry associated to two of the three $SO(3)$ generators, we expect two Goldstone modes [3]. The residual $SO(2)$ symmetry of rotations about $\boldsymbol{n}_0$ plays two important roles in organizing these modes: on the one hand, it ensures that the two modes have identical speeds; on the other, it allows one to ascribe quantum numbers, termed "helicities" — given by the projection of the spin along $\boldsymbol{n}_0$ — to their quanta.

As mentioned earlier, the ground state of the triangular Heisenberg antiferromagnet appears to fully break the $SO(3)$ symmetry, and indeed a standard spin-wave calculation finds three gapless Goldstone modes in the spectrum as expected from this. Since there is no apparent residual symmetry, one might expect that there is neither any particular degeneracy in the mode speeds, nor a sensible quantum number to label them. Remarkably, two of the three modes have an identical dispersion, suggesting that there is something missing in the conventional treatment.

To understand what is going on, we need to think harder about the nature of the corresponding NLSM. For the case at hand, the magnetic order parameter can be captured by a triplet of unit vectors $\{\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3\}$, each pointing in the direction of the magnetization on one of the three sublattices. This triplet can be rotated collectively without altering the internal structure — exactly as for a rigid body. Since the orientation of a rigid body can be parametrized as an $SO(3)$ rotation matrix O_{ab}, one way to describe this problem is in terms of an NLSM for such matrices [5].

Two of the present authors recently proposed an elegant alternative, in terms of a "spin frame" consisting of two linear combinations of $\boldsymbol{m}_a$ and their cross product:

$$\begin{aligned}
\boldsymbol{n}_x &= \frac{(\boldsymbol{m}_2 - \boldsymbol{m}_1)}{\sqrt{3}}; \quad \boldsymbol{n}_y = \frac{(2\boldsymbol{m}_3 - \boldsymbol{m}_2 - \boldsymbol{m}_1)}{3}; \\
\boldsymbol{n}_z &= \boldsymbol{n}_x \times \boldsymbol{n}_y = \frac{2}{3\sqrt{3}}(\boldsymbol{m}_1 \times \boldsymbol{m}_2 + \boldsymbol{m}_2 \times \boldsymbol{m}_3 + \boldsymbol{m}_3 \times \boldsymbol{m}_1).
\end{aligned} \tag{2}$$

In the ground state where $\boldsymbol{m}_1 + \boldsymbol{m}_2 + \boldsymbol{m}_3 = 0$, the $\boldsymbol{n}_a$ form an orthonormal set; longwavelength fluctuations are then described by a NLSM for the spin-frame fields: upto topological terms, we have

$$\mathcal{L} = \frac{\rho}{4} \sum_{a=x,y,z} \partial_t \boldsymbol{n}_a \cdot \partial_t \boldsymbol{n}_a - \frac{\mu}{2} \sum_{i,j=x,y} \partial_i \boldsymbol{n}_j \cdot \partial_i \boldsymbol{n}_j \tag{3}$$

note the anisotropy between z and x, y. A detailed calculation of fluctuations around a broken symmetry state $\{\boldsymbol{n}_x^0, \boldsymbol{n}_y^0, \boldsymbol{n}_z^0\}$ in Eq. (3) reveals the aforementioned surprise: the existence of three modes, two of which have the same dispersion due to an $SO(2)$ symmetry of the low-energy Goldstone action.

An important clue is gleaned by examining the relationship between the spin frame and the $SO(3)$ rotation matrix formulations, which are ultimately equivalent. This can be done by introducing a global frame $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$, and writing $\boldsymbol{n}_b = \boldsymbol{e}_a O_{ab}$, or equivalently,

$\boldsymbol{e}_a = O_{ab}\boldsymbol{n}_b$. One can view the two triples $\{\boldsymbol{e}_a\}$ and $\{\boldsymbol{n}_b\}$ as corresponding to the two ways of parametrizing a general $SO(3)$ rotation matrix: either through a series of extrinsic rotations about axes fixed in the global coordinate system $\{\boldsymbol{e}_a\}$, or via intrinsic or "Euler" rotations about body-fixed axes $\{\boldsymbol{n}_b\}$.

The rotation matrix O captures the relative orientation between the frames: we can write

$$O_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{n}_b, \tag{4}$$

so that the bth column of O contains the components of $\boldsymbol{n}_b$ in the global frame, while the ath row of O contains the components of $\boldsymbol{e}_a$ in the body frame, respectively: in equations, $(\boldsymbol{n}_b)_a = \boldsymbol{e}_a \cdot \boldsymbol{n}_b = (\boldsymbol{e}_a)_b$. Since an extrinsic rotation described by a rotation matrix $L \in SO(3)$ is *defined* with respect to the global frame, it transforms the *body-fixed* vectors componentwise, $(\boldsymbol{n}_b)_a \mapsto L_{ac}(\boldsymbol{n}_b)_c$. Conversely, an intrinsic rotation described by a rotation matrix $R \in SO(3)$ *defined* with respect to the body-fixed frame transforms the global basis vectors as $(\boldsymbol{e}_a)_b \mapsto R_{bc}(\boldsymbol{e}_a)_c$. The action of extrinsic and intrinsic rotations is more transparently written in terms of O, where they respectively correspond to left and right multiplication; they therefore commute with each other, and can act simultaneously, viz.

$$O \mapsto LOR^T. \tag{5}$$

The relevant symmetry of O under these transformations is that of the *chiral group* $G = SO(3)_L \times SO(3)_R$ (named in analogy to the chiral group in particle physics, rather than because there is any intrinsic handedness present). Technically, the discussion above makes no use of the specific energetics of the microscopic theory; it turns out the symmetry of the triangular Heisenberg antiferromagnet is slightly lower, but it is useful to proceed assuming the higher symmetry.

Identifying this enhanced "two-sided" action of symmetries in noncollinear antiferromagnets is a central insight of the work [6]. Much of the balance of the paper flows smoothly from it. An elegant trick is to use the isomorphism between the Lie algebras of $SO(3) \times SO(3)$ and $SO(4)$ (familiar to anyone who has worked through Pauli's solution of the hydrogen spectrum [7]) to move to a description in terms of an NLSM in terms of an $O(4)$ order parameter unit vector $q \in S^3$ (with antipodal points identified). While the direct transcription of Eq. (3) in terms of q has lower symmetry (reflecting the fact that the actual symmetry in the triangular Heisenberg magnet symmetry is not quite $SO(3)_L \times SO(3)_R$ but is somewhat smaller), an initial simplification is to consider a fully $SO(4)$ symmetric problem, corresponding to what is known as the *principal chiral model* (PCM) [8].

Within the PCM, rather than working with the 'left' and 'right' symmetries $SO(3)_{L,R}$, it turns out to be more natural to work with the 'vector' and 'axial' components of the chiral group, $SO(3)_V = SO(3)_L \times SO(3)_{R=L^{-1}}$ and $SO(3)_A = SO(3)_L \times SO(3)_{R=L}$. The broken symmetry ground state breaks $G = SO(3)_L \times SO(3)_R$ down to $SO(3)_V$; in the matrix representation, such transformations correspond to $O \mapsto LOL^{-1}$, which describes the residual symmetry. Since $SO(3)_A$ is fully broken, we recover the correct count of 3 Goldstone modes, but the $SO(3)_V$ now constrains all these three to have identical speeds, and permits a labelling in terms of "isospin" quantum numbers, $-1, 0, 1$ (in units of $\hbar$) corresponding to the eigenvalue of a single component of the isospin.

The enhanced symmetry of the PCM corresponds to a freedom to choose the isospin axis, which is the physical origin of the Goldstone mode degeneracy. In the original physical problem — the triangular Heisenberg model — $\mathcal{L}$ in is only invariant under rotations about $\boldsymbol{n}_z$, so that the $SO(3)_R$ symmetry of body-center axis rotations is reduced to $SO(2)_{R,3}$ corresponding to rotations about the third body-fixed axis. This translates into a reduced $SO(2)_{V,3} \subset SO(3)_V$ residual symmetry, corresponding to matrix conjugation by an axis perpendicular to the spin plane in the ground state (and hinting at the importance of co-*planarity* to the Goldstone description). In more physical terms, the emergent symmetry is a specific combination of joint rotations about the global axis $\boldsymbol{e}_z$ and the local spin-frame axis $\boldsymbol{n}_z$. As a consequence of this symmetry structure, the Goldstone branches exhibit lower symmetry than the PCM with only two of the three modes having the same speed. The $SO(2)_{V,3}$ invariant Goldstone modes correspond to circularly polarized magnons, that form a ± 1 isospin doublet upon quantization [11].

The approach developed by the authors thus resolves the immediate puzzle of the triangular Heisenberg antiferromagnet while also introducing an elegant new framework for exploring other frustrated magnetic systems using field-theoretic tools. It is not too difficult to imagine a wide range of applications to other more complex orders, including those that are non-coplanar, even to systems that exhibit a greater degree of frustration (e.g. the famed kagomé Heisenberg antiferromagnet) or involve quenched disorder (e.g., spin glasses). In such cases, understanding the connection to hydrodynamic approaches [9, 10] may offer further insight. Overarching lessons from this work are that beautiful surprises remain in even as well-trodden a subject as symmetry breaking, and that models devised with very different motivations in the particle physics setting continue to find fruitful application to concrete problems of condensed matter.

The correspondent would like to dedicate this commentary to the memory of Prof. Assa Auerbach, from whose classic book he first learnt many aspects of quantum magnetism, and who passed away suddenly earlier this year.

References

[1] See, e.g., A. Auerbach, *Interacting Electrons and Quantum Magnetism*, Graduate Texts in Contemporary Physics. Springer New York, 1998.

[2] A proper quantum treatment should also include a possible topological term that famously distinguishes integer and half-odd-integer spins in one spatial dimension, but this is not terribly important for our present discussion.

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[10] A. F. Andreev and V. I. Marchenko, Symmetry and the macroscopic dynamics of magnetic materials, Sov. Phys. Uspekhi **23**, 21 (1980).

[11] In fact, there is a small subtlety: $SO(2)$ is Abelian and has only one-dimensional irreducible representations. However, the addition of a symmetry comprising time-reversal composed with inversion (or C_2 rotation) combines with $SO(2)$ to lead to a degenerate doublet. I thank the authors for a helpful correspondence on this point.