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Classical and quantum glasses based on expanders

1. **Topological Quantum Spin Glass Order and its realization in qLDPC codes**
   Authors: Benedikt Placke, Tibor Rakovszky, Nikolas P. Breuckmann, Vedika Khemani
   arxiv:2412.13248
2. **Expansion creates spin-glass order in finite-connectivity models: a rigorous and intuitive approach from the theory of LDPC codes**
   Authors: Benedikt Placke, Grace M. Sommers, Nikolas P. Breuckmann, Tibor Rakovszky, Vedika Khemani
   arXiv:2507.13342

Recommended with a Commentary by Roderich Moessner, MPI-PKS Dresden

General background— Slow dynamics and the failure of many-body systems to equilibrate present a perennial challenge to many body physics. These appear in many guises: jamming, glasses, spin glasses, many-body localisation. The relationship between these (and in some instances, their very existence) are not entirely clear. At the same time, the scales involved in terminating any transient behaviour realising these phenomena can be astronomical, so that the capacity of numerics to provide definitive answers can be limited, and at any rate, timescales in experiment such as the lifetime of the system under consideration (or indeed, that of the research grant) or the patience of a graduate student can be considerably shorter than any crossover time. All the more, these are well-studied problems, and much experimental effort has been devoted to them.

A theoretical connection between spin glasses and error correction was already made long ago [1, 2, 3]. The motivation for studying such systems now is perhaps as big as it has ever been. In particular, there is massive interest in the context of the need to error-correct quantum computational devices in order to reach an error threshold of collective logical qubits which appears unattainable for individual physical qubits. In this context, low density parity check (LDPC) codes are playing an important role. (Kitaev's toric code perhaps is the instance of an LDPC code most familiar to the quantum many-body physics community).

The (spin) glass literature may not be the most easily penetrable to the newcomer, in part due to the heterogeneity of phenomena, models, and descriptive frameworks. Spin glass models do come in several flavours, with salient differences in their degree of locality. This includes the completely non-local random energy model [4] with a number of random couplings growing exponentially with the number of degrees of freedom; the celebrated Sherrington-Kirkpatrick spin glass with infinite range pair-wise interactions [5], and its cousin, the p-spin glass [6] where either all, or a set of randomly chosen 'dilute', groups of p spins interact with one another. The Edwards-Anderson spin glass model with disordered purely nearest-neighbour couplings on a d-dimensional lattice is the version with the most local interactions [7].

From these studies, it is clear that dimensionality plays an important role: the Sherrington-Kirkpatrick model can be solved exactly and exhibits features (replica symmetry breaking) which may or may not persist to low dimension. The fact that the EA low-temperature phase is also called a spin glass does not necessarily imply it has the same properties as the SK one.

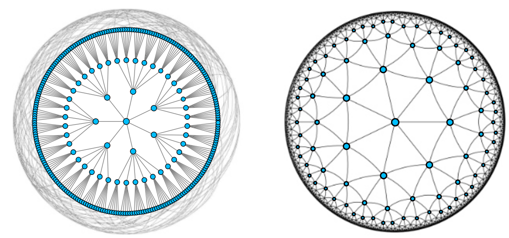

Figure 1: Expander graphs: locally tree-like graph (left) and hypberolic tesselation (right).

The highlighted preprints grow out of this interest in studying dense and robust encoding of information. In statistical physics language, what one is after in an efficient code is a Hamiltonian defining an energy landscape that has a large number of global minima (the code words) separated by high energy barriers (to avoid fluctuations of whatever origin inducing transitions between them) surmounting which requires a large number of spin flips .

For a given spin model/code, the quantities representing these properties are collated in a triplet of numbers $[n, k, d]$: n is the number of degrees of freedom (spins/(qu)bits) of the system under consideration; 2^k is the number of ground states/code words, amounting to k bits of information; and, crucially, the distance d is the minimal number of degrees of freedom that need to be flipped to move from one ground state to another.

Classical glassiness— Now, in finite dimensional systems with short-range interactions, one cannot arrange both a finite ratio of encoded information density, k/n, and a distance d which diverges with n. A way to realise these desiderata togther is to use expander graphs. Their definition is, loosely speaking, that any piece cut out of an expander graph has a surface of comparable size to its volume: for any (sufficiently small) subgraph of an expander, the number of neighbours outside the subgraph is proportional to the number of sites inside it. This makes expanders 'infinite-dimensional' objects, as by contrast, the ratio of the surface of a finite dimensional hypercube to its volume is $\sim L^{d-1}/L^d \to 0$ for $L \to \infty$.

Expander graphs come in various familiar forms, Fig. 1, including locally tree-like graphs (such as Erdos-Renyi or regular random graphs), or tesselations of hyperbolic plane. Most readers will be familiar with issues such as the absence of *gapless* Goldstone modes on these graphs, or the intricacies of distinguishing between Cayley trees and Bethe lattices on account of the influence of boundary conditions on their bulk properties.

Low-density parity-check codes (LDPC) which take centre stage in the highlighted works are basically Ising models with an extensive number of terms involving products of several spins, $\prod_{i \in C_\alpha} \sigma_i$. The sets C_α define the expander interaction (hyper-)graph, with each C_α

connected to its σ_i, and each σ_i connected to the C_α of the checks it appears in. The abovementioned diluted (i.e., not all p-tuples appear in the Hamiltonian) p-spin glass with ’ferromagnetic’ couplings is an instance of an LDPC.

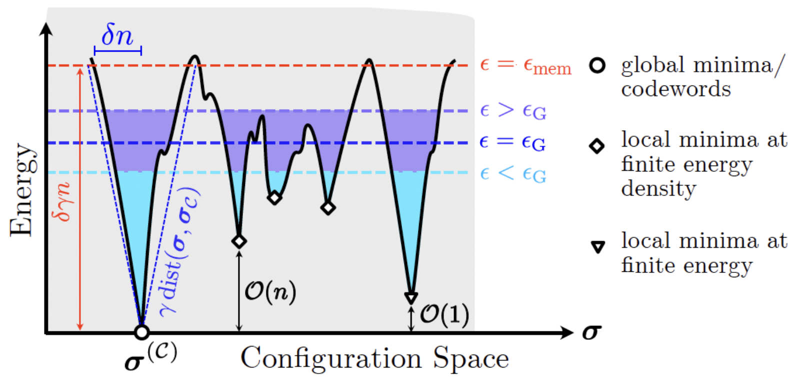

Figure 2: Energy landscape around one code-word (one of the global minima) of an LDPC.

The LDPC can be treated efficiently numerically, allowing for numerics up to tens of thousands of spins in this work. But its main attraction is that this work provides rigorous statements about the statistical mechanics of such models, which is often not easy for glassy systems.

In particular, the work identifies LDPC codes/diluted p-spin glasses with a sufficiently high value of p (and other conditions detailed in the work) as having the properties sketched Fig. 3. There are extensively (a number exponential in system size n) many global ground states. These have extensively high energy barriers between them. In addition, further local minima exist, both at non-zero energy but vanishing energy density, and at non-zero energy density. The latter come with lower but still extensive energy barriers. It is shown that the energy barriers ‘survive’ to become extensive free energy barriers between Gibbs states at finite temperatures. This can then be used to show that at sufficiently low temperature, the system spontaneously chooses to enter one of the exponentially numerous Gibbs states corresponding to the local and global minima, and thus enters a glassy phase. This phase is essentially the one familiar from one-step replica symmetry breaking, in the language of which many similar results have been obtained with a lower level of rigour.

Topological quantum glassiness— The other highlighted work, on quantum spin glasses, identifies a route from this classical energy landscape into a quantum version of glassiness. This result is then directly relevant to the question whether a quantum LDPC can encode a finite density of information robustly. This work proceeds in several steps, and the following gives a brief outline of the central ingredients.

First, to turn a classical into a quantum LDPC, one now employs terms like the above $\prod_{i\in C_\alpha}\sigma_i$ containing different spin components. This idea will be familiar from the case of Kitaev’s toric code, with its product of the four σ^z’s around a square plaquette (so-called Z-stabilisers), and the four σ^x’s emanating from a site of the square lattice (X-stabilisers). However, the generalisation of this construction to an expander setting is not purely mechanical, i.e. one does not simply add identical/similar terms in X- and Z-bases. Rather, one takes a pair of classical codes described above and forms a so-called hypergraph product out of these [8], as sketched impressionalistically in Fig. 3. Such a construction then ensures the existence of an extensive number of global minima, just as above, with a code distance and energy barriers growing as $d \sim \sqrt{n}$ (rather than n as in the above classical model).

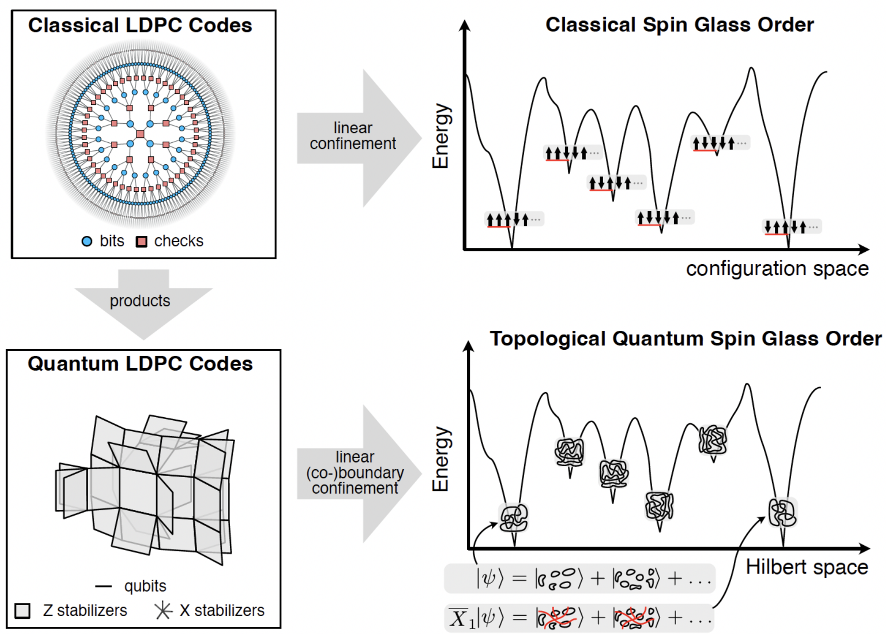

Figure 3: Right: Energy landscape of glasses based on classical and quantum LDPC. Left: interaction hypergraph of classical LDPC (top) and quantum LDPC, the latter arising as a Cartesian (or box) product of two classical LDPCs.

The properties of the quantum ground states are remarkable in that they express what may be termed a form of topological order–all of them exhibit long-range entanglement. That is to say that, like the case of the ground states of the toric code, the ground states are locally indistinguishable and can only be constructed from a product state by a quantum circuit of depth at least $\log d$. In this sense, a quantum code constructed from the ground states of these quantum LDPC incorporates the idea of topologically protected quantum information. In addition, the authors show that the local minima described above evolve into Gibbs states which exhibit at least some degree of long-range entanglement, i.e. none of these Gibbs states can be represented by a mixed states of short-range entangled states either.

Unlike in the case of simple transverse field Ising models, these ground states are therefore ‘not just’ relatively simple zero temperature extensions of classical spin states. Rather, their long-range entanglement is a constitutive feature, as reflected in the moniker ‘topological quantum spin glass’.

I hope this commentary has managed to illustrate how the reinvigorated confluence of information theory and statistical mechanics can lead to interesting developments, which are a long way from having run their course.

References

[1] N. Sourlas, ”Spin-glass models as error-correcting codes”, Nature **339**, 693-695 (1989).

[2] H. Nishimori, ”Statistical Physics of Spin Glasses and Information Processing” (Oxford University Press, 2001).

[3] M. Mezard, A. Montanari, ”Information, Physics, and Computation” (Oxford University Press, 2009).

[4] B. Derrida, "Random-Energy Model: Limit of a Family of Disordered Models", Physical Review Letters **45**, 79-82 (1980)

[5] D. Sherrington, S. Kirkpatrick, "Solvable Model of a Spin-Glass", Physical Review Letters **35**, 1792-1796 (1975)

[6] D.J. Gross, M. Mézard, "The simplest spin glass", Nuclear Physics **B 240(4)**, 431–452 (1984).

[7] S.F. Edwards, P.W. Anderson, "Theory of spin glasses", J. Phys. F: Met. Phys. **5**, 965 (1975).

[8] J.-P. Tillich, G. Zemor, "Quantum ldpc codes with positive rate and minimum distance proportional to $n^{1/2}$", in 2009 IEEE International Symposium on Information Theory (2009) pp. 799–803.